Description of Classical and Quantum Anisotropic Cosmological Models in Terms of Raychaudhuri and Wheeler-DeWitt's Equations

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Introduction

Anisotropic cosmological models with perfect fluid are considered. Observational cosmological parameters have been related to rotation and shear. The results are compared with recent upper bounds on the Universe's global rotation and Hubble's parameter anisotropy. The Lagrangian for anisotropic cosmological models has been obtained from Raychaudhuri's equation providing the acceleration divergence vanishes. Wheeler-DeWitt's equation is derived following the standard Hamiltonian formalism. Taking account rotation and shear, the Universe's birth probabilities (penetration factors) have been calculated for a flat Universe filled with dust and de Sitter's vacuum.

Cosmological Fluid

We shall follow a so-called hydrodynamic approach, in which matter in the Universe is considered to be a perfect fluid.

The cosmological fluid motion is described in general relativity by Raychaudhuri's equation

$$\dot{\theta}+\tfrac{1}{3}\theta^2-A^i_{;i}+2(\sigma^2-\omega^2)=-\frac{4\pi G}{c^2}(\varepsilon+3p) \qquad (1)$$

where θ is the expansion scalar, A^i is the 4-acceleration, σ is the shear scalar, ω is the rotation scalar, ε is the energy density, p is the pressure.

The space-time metric reads

$$ds^2=(N^2-N_\alpha N^\alpha)dt^2-2N_\alpha dt dx^\alpha-\gamma_{\alpha\beta}dx^\alpha dx^\beta \quad (\alpha,\beta=1,2,3) \qquad (2)$$

where N is the lapse function, $N_\alpha=cg_{0\alpha}$ is the shift function and the metric cross term $g_{0\alpha}$ satisfies the relation $g^{00}=1-g_{0\alpha}g^{0\alpha}$, $\gamma_{11}\gamma_{22}\gamma_{33}=a^3$, $a(t)$ is the scale factor. The quantities θ, A^i, σ, ω read $\theta = u^i_{;i}$, $A^i = u^i_{;k} u^k$, $\omega^2 = \frac{1}{2}\omega_{ik}\omega^{ik}$, $\sigma^2 = \frac{1}{2}\sigma_{ik}\sigma^{ik}$ where u_i is the 4-velocity. The orthogonality conditions have the form $u_i \omega_{ik} = 0$, $u^i \sigma_{ik} = 0$, $u^i A_i = 0$.

From the conservation law $T^{ik}_{;k} = 0$ for the energy-momentum tensor

$$T^{ik} = (p+\varepsilon)u^i u^k - p g^{ik}, \tag{3}$$

providing $\theta = \dfrac{3\dot{a}}{a}$, which is valid if $\sigma_{\alpha\beta} n^\alpha n^\beta = 0$, where $n^\alpha n_\alpha = 1$, we obtain an expression for the 4-acceleration divergence scalar as follows

$$A^i_{;i} = w(1 - g^{00})(\dot{\theta} + \theta^2), \;\; g^{00} = const \tag{4}$$

where $1 - g^{00}$ characterizes a deviation from Friedmann's geometry. The equation of state $p = w\varepsilon$ where $w = const$.

Cosmological Parameters

Introduce cosmological parameters:
Hubble's parameter

$$H = \frac{\dot{a}}{a} \, , \tag{5}$$

the average density in critical density units

$$\Omega = \frac{8\pi G \varepsilon}{3c^2 H^2} \tag{6}$$

and the deceleration parameter

$$q = -\frac{\ddot{a} a}{\dot{a}^2} \, . \tag{7}$$

Two-Component Model

Consider the cosmological matter to be a two-component medium consisting of de Sitter's vacuum and dust with the equations of state $p=-\varepsilon$ and $p=0$, the average density Ω_Λ and Ω_m respectively. For de Sitter's vacuum we have $\sigma=\omega=A^i=0,\quad g^{00}=1$.

The deceleration parameter derived from Raychaudhuri's equation takes the form

$$q=-\Omega+\frac{3\Omega_m}{2}+\frac{2}{3}\frac{\sigma^2-\omega^2}{H^2} \qquad (8)$$

where $\Omega=\Omega_\Lambda+\Omega_m$.

From CMBR observations, it follows that $\Omega_\Lambda = 0.7,\ \Omega_m = 0.3$.

Since $\frac{\sigma}{H} < \frac{\omega}{H} < \frac{\Delta T}{T} < 10^{-5}$, the anisotropy related to shear and rotation almost does not affect the deceleration parameter. The deceleration parameter error estimated from Hubble's parameter anisotropy

$$\Delta q = 3\left(\Omega - \Omega_m\right)\frac{\Delta H}{H_0} \tag{9}$$

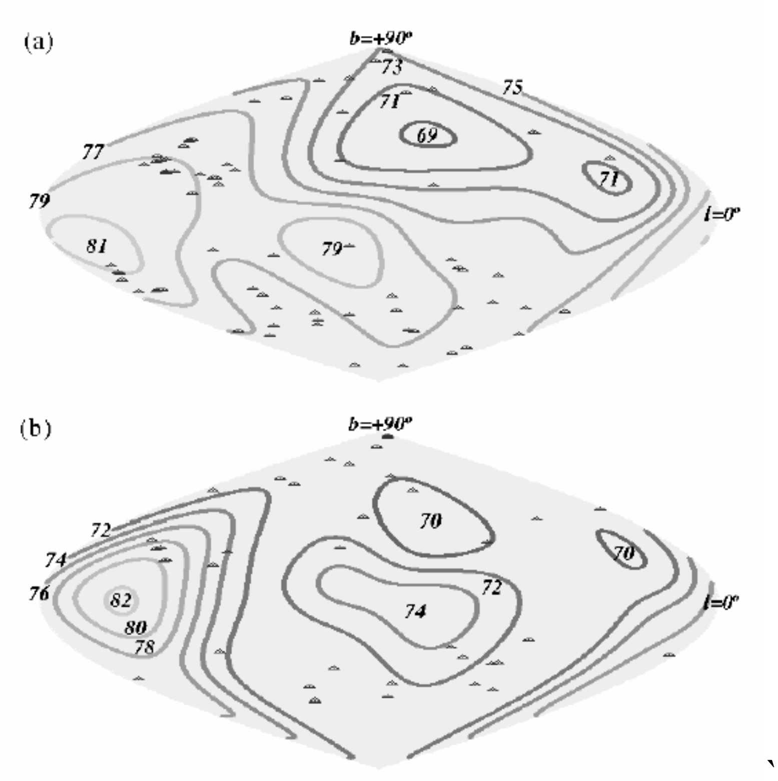
as $\Delta q < 0.45$ for $\frac{\Delta H}{H_0} < 0.2$ is in agreement with observations.

References

1. *G. Hütsi* “Power spectrum of the SDSS luminous red galaxies: constrains on cosmological parameters”
<astro-ph/0604129v2 (13 Dec 2006)>

2. *M. L. McClure, C. C. Dyer* “Anisotropy in the Hubble constant as observed in the HST Extragalactic Distance Scale Key Project results”
<astro-ph/0703556v1 (21 Mar 2007)>.

Hubble constant contour maps in $km/s \cdot Mpc$ units



Hamiltonian Formalism

The Lagrangian for perfect fluid is given by the formula

$$L = \frac{3p - \varepsilon}{2} N a^3 . \tag{10}$$

For dust and de Sitter's vacuum $A^i_{;i} = 0$. Raychaudhuri's equation in this case takes the form

$$\frac{\ddot{a} c^2}{N^2} = -\frac{4\pi G}{3c}\left(\varepsilon + 3p\right) a + \frac{2}{3}\left(\omega^2 - \sigma^2\right). \tag{11}$$

Deriving ε and p from (11), we obtain

$$L=\frac{a\dot{a}^2}{2N}-\frac{kNa}{2}+\frac{4\pi G\varepsilon}{3c^4}Na^3+\frac{2Na}{3c^2}\int\left(\omega^2-\sigma^2\right)ada\,. \tag{12}$$

The Hamiltonian $H=\frac{\partial L}{\partial \dot{a}}\dot{a}-L$ reduces to

$$H=N\left[\frac{p_a^2}{2a}+\frac{ka}{2}-\frac{4\pi G\varepsilon}{3c^4}a^3-\frac{2}{3c^2}a\int\left(\omega^2-\sigma^2\right)ada\right], \tag{13}$$

where $p_a=\frac{a\dot{a}}{N}$. Since $\frac{\partial L}{\partial \dot{N}}=0$, $\frac{\partial H}{\partial N}=0$, and we have a Hamiltonian constraint

$$\frac{p_a^2}{2a}+\frac{ka}{2}-\frac{4\pi G\varepsilon}{3c^4}a^3-\frac{2}{3c^2}a\int\left(\omega^2-\sigma^2\right)ada=0\,. \tag{14}$$

Quantum Cosmology

Introducing the operator $\hat{p}_a = \frac{l_{pl}^2}{i}\frac{d}{da}$, which acts on the Universe's wave function ψ, from the Hamiltonian constraint we obtain Wheeler-DeWitt's equation

$$\frac{d^2\psi}{da^2} - V(a)\psi = 0, \tag{15}$$

which describes the early Universe in framework of quantum cosmology, where the potential

$$V(a) = \frac{1}{l_{pl}^4}\left[ka^2 - \frac{8\pi G\varepsilon}{3c^4}a^4 - \frac{4}{3c^2}a^2\int\left(\omega^2 - \sigma^2\right)ada\right]. \tag{16}$$

$$\varepsilon = \varepsilon_0 \sum_{n=0,3} B_n \left(\frac{r_0}{a} \right)^n, \quad B_0 + B_3 = 1. \tag{17}$$

Here r_0 and ε_0 are de Sitter's radius and energy density respectively.

For dust we have

$$\omega = \frac{Jc^2}{\varepsilon a^5} \tag{18}$$

(angular momentum conservation law)

$$\sigma = \frac{\Sigma}{a^3}. \tag{19}$$

For de Sitter's vacuum $\omega = \sigma = 0$ and $g^{00} = 1$.

The potential may be represented in the form

$$V(a) = f_1 a^2 + f_2 a^4 + f_3 a + f_4 + f_5 a^{-2} \tag{20}$$

where

$$f_1 = \frac{k}{l_{pl}^4}(k=0,\pm 1), f_2 = -\frac{B_0}{r_0^2(6-5g^{00})l_{pl}^4}, f_3 = -\frac{B_3 r_0}{l_{pl}^4}, f_4 = \frac{2c^2 J_3^2}{3\varepsilon_0^2 l_{pl}^4 B_3^2 r_0^6}, f_5 = -\frac{\Sigma_3^2}{3c^2 l_{pl}^4} .$$

The Universe's birth probability (penetration factor) is given by Gamow's formula that near the potential maximum a_m takes the form

$$D = \exp\left(\frac{\pi\sqrt{2}V(a_m)}{\sqrt{-V''(a_m)}}\right) \tag{21}$$

where $V'(a_m) = 0$. For a flat model (k=0) we have

$$a_m = \sqrt[3]{-\frac{f_3}{8f_2} + \sqrt{\frac{1}{64}\left(\frac{f_3}{f_2}\right)^2 + \frac{f_5}{2f_2}}} . \tag{22}$$

Evaluating $V_m = V(a_m)$ and $V_m'' = V''(a_m)$, we obtain

$$D = \exp\left(-\pi\sqrt{2}\frac{f_2 a_m^4 + f_3 a_m + f_5 a_m^{-2} + f_4}{\sqrt{-12 f_2 a_m^2 - 6 f_5 a_m^{-4}}}\right). \tag{23}$$

Thus, de Sitter's vacuum, dust and shear increase D, and rotation decreases it.

Conclusions

1. The recent observational data reinterpreted in the framework of anisotropic cosmological models allow one to derive some spatial anisotropy parameters.
2. In the two-component model consisting of de Sitter's vacuum and dust, the deceleration parameter errors may be due to Hubble's constant anisotropy.
3. The Hamiltonian constraint has been derived from Raychaudhuri's equation providing the acceleration divergence vanishes.
4. Wheeler-DeWitt's equation was obtained from the Hamiltonian constraint.
5. The penetration factor has been calculated for a flat model with dust and de Sitter's vacuum taking account rotation and shear.