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Structure of Hypercomplex Units and Exotic Numbers as Sections of Bi-Quaternions

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A survey of all families of hypercomplex (HC-) numbers is suggested with emphasis on exotic sets. Systematic description of variety of representations of HC-units is given, and interior structure of the units is studied. Elementary math objects constituting the structure are demonstrated to possess variously algebraic, geometric and physical properties, being eigenfuctions of HC-vector operators, ideals of idempotent matrices, dyads (Lame coefficients) linking two 2-dimensional surfaces, projectors of matrix-vectors onto given axis, and spinors. It is also shown that full set of bi-quaternion numbers comprises as special cases real, complex, quaternion numbers and as well exotic sets split-complex and dual numbers. In particular a HC-unit of double numbers is found to be represented by a Pauli-type matrix, and a simple formula for null-modulus HC-unit of dual numbers is indicated.

1. INTRODUCTION. EXOTIC NUMBERS WITH ASSOCIATIVE MULTIPLICATION

Pioneering works of Pauly and Dirac demonstrated extremely successive use of "not ordinary" numbers to describe hardly conceived physical manifestations, e.g., an elementary particle's spin. Since, the interest towards involved mathematics was growing, and research papers in domain of theoretical physics of beginning of XXI centaury already demonstrate nearly global increase of interest to different types of the so called hypercomplex numbers, and to their use in modeling of physical phenomena. All types of such numbers obeying associative multiplication rules are analyzed below.

Apart from three types of numbers forming "good" associative (in multiplication) sets, real and complex numbers (fields), and quaternions (non-commutative ring), there are also three different sets composed of as well multiplicatively associative but "not as good" (exotic) numbers, split-complex and dual numbers, and bi-quaternions. Leaving aside well known "good" sets, shortly recall properties of less famous exotic numbers, which none-the-less start attracting more attention and use in description of physical phenomena. In many papers the exotic numbers are regarded as self-meaning closed sets deserving separate investigations; this position finds reflection in this introducing paragraph. But the paper's goal is to show that each number set enumerated above manifests itself just as a layer (or a section) of a unique generic set.

Split-complex numbers (tessarines, motors; perplex, double, hyperbolic numbers)[1–4] $s = x \cdot 1 + y \cdot j$ (or simply $s = x + jy$, x, y being arbitrary real numbers) similarly to complex numbers are built onto two basic unites, 1, j, but squares of the both are equal to a real unity $1^2 = 1$, $j^2 = 1$, while $1 \cdot j = j \cdot 1 = j$. This set of numbers admits majority of actions appropriate to complex numbers: addition, commutative, associative and distributive (over addition) multiplication, conjugation $s^* = x - jy$, and a type of a "norm" $\|s\| = s \cdot s^* = \sqrt{(x+jy)(x-jy)} = \sqrt{x^2 - y^2}$. Expression of the square "norm" $\|s\|^2 = x^2 - y^2$ repeating equation of a hyperbola prompts to name the numbers hyperbolic; this sometimes pushes the authors to associate the numbers with 2-dimensional (2D) Minkowsky metric[4] and 2D Lorents transformations (the association being somewhat artificial). For $\|s\| = 1$ one discovers an analogue of the Euler formula $e^{j\psi} = \cosh\psi + j\sinh\psi$, where$x = \cosh\psi$, ψ is a hyperbolic parameter ("angle"). But it is evident that the "norm" can be zero what means that in general the numbers are non-invertible; hence division of their algebra is defected. Being two-valued the split-complex numbers can be geometrically represented as vector onto a surface, usually plane; then two simple conjugate vector $e = (1+j)/2$, $e^* = (1-j)/2$ are "orthogonal" $ee^* = 0$ and idempotent $e^N = e$, $e^{*N} = e^*$ (N is an integer), and they form an orthogonal (or diagonal) basis for any split-complex number $s = (x+y)\,e + (x-y)\,e^*$. A number $s = x + jy$ admits the most simple 2×2-matrix representation

$$s = \begin{pmatrix} x & y \\ y & x \end{pmatrix} \tag{1}$$

modulus given by the matrix determinant $\|s\| = \det s$. It is usually said that the representation (1) is not unique due to variety of 2×2-matrix forms fit for description of split-complex numbers; this important observation will be discussed below.

Dual numbers (parabolic numbers) $d = x \cdot 1 + y \cdot \varepsilon$ (or simply $d = x + \varepsilon y$, x, y being arbitrary real numbers)[5–7] are also an extension of real numbers built onto two units: 1, ε, the first unit

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being a real (and ordinary scalar) one $1^2 = 1$, the second one satisfying the nilpotent square-rule $\varepsilon^2 = 0$, while $1 \cdot \varepsilon = \varepsilon \cdot 1 = \varepsilon$. The set of dual numbers also admit actions appropriate to complex numbers: addition, commutative, associative and distributive (over addition) multiplication, conjugation $d^* = x - \varepsilon y$; the norm is $\|d\| = d \cdot d^* = \sqrt{(x+\varepsilon y)(x-\varepsilon y)} = \sqrt{x^2} = x$, i.e., real part of the dual number. This means that norm of the number $d = \varepsilon y$ vanishes; all such nilpotent members of the dual-number algebra defect its division. Due to nilpotent power property the unit ε can be used similarly to infinitesimal parameters; this cuts Tailor series of a function within only two members. Thus, a dual number with unit norm ($x = 1$) is nothing but full series development of the exponent $e^{\varepsilon y} = 1 + \varepsilon y$, what represents the analogue of the Euler formula where component y plays the role of angular parameter. The exponent of the form $e^{\varepsilon t}$ (t is a parameter) performs "parabolic rotation" of a dual number $d = x + \varepsilon y$ resulting in the translation $e^{\varepsilon t} d = x + \varepsilon(y + xt)$ what sometimes is used to link dual numbers with Galilean group. The simplest 2×2-matrix representations of a dual number are normally suggested in math literature as

$$d^{\uparrow} = \begin{pmatrix} x & y \\ 0 & x \end{pmatrix}, \quad \text{or} \quad d^{\downarrow} = \begin{pmatrix} x & 0 \\ y & x \end{pmatrix} \tag{2}$$

it is shown in Section 3 that these representations are not unique.

Finely, bi-quaternions[8–10] are numbers given in the form $b = x \cdot 1 + y_k \cdot \mathbf{q}_k$ (or simply $b = x + y_k \mathbf{q}_k$) where small Latin indices are 3-dimensional (3D): $k, l, m, n = 1, 2, 3$ (summation in repeating indices is implied), x, y_k (scalar and vector components) are complex numbers, and 1, $\mathbf{q}_k$ are four units generating multiplication table of quaternions

$$1\,\mathbf{q}_k = \mathbf{q}_k 1, \quad \mathbf{q}_k \mathbf{q}_m = -\delta_{km} + \varepsilon_{kmn} \mathbf{q}_n \tag{2a}$$

with δ_{km}, ε_{kmn} being Kronecker and Levi-Civita symbols. The set of the numbers admit algebraic addition, commutative, associative and distributive (over addition) multiplication. Biquaternions can be conjugated as ordinary quaternion $\bar{b} = x - y_k \mathbf{q}_k$. But this operation doesn't lead to satisfactory definition of the norm since the usual product $b\,\bar{b} = (x + y_k \mathbf{q}_k)(x - y_m \mathbf{q}_m) = x^2 - y_k y_k$ can be neither a positive, nor a real number; it is also clear that among bi-quaternions zero divisors are present. Differently from other above regarded numbers two extra types of conjugation can be applied to bi-quaternions: complex conjugation $b^* = x^* + y_k^* \mathbf{q}_k$ and mixed "hermitian" conjugation[10] $b^{\times} = \bar{b}^* = x^* - y_k^* \mathbf{q}_k$, but they don't cure the norm disease. Nonetheless if vector part consists of a bi-quaternion consist of orthogonal real and imaginary vectors $y_k = w_k + i z_k$, $w_k z_k = 0$, (w_k, z_k are non-zero real numbers) then this bi-quaternion has a definite norm[9] $\|b\| = \sqrt{b\,\bar{b}} = \sqrt{[x + (w_k + i z_k)\mathbf{q}_k][x - (w_m + i z_m)\mathbf{q}_m]} = \sqrt{x^2 + w_k w_k - z_k z_k}$, but subset of these numbers with definable norm as well obviously contains zero divisors. If $w_k = 0$, then square norm of the number is $\|b\|^2 = x^2 - z_k z_k$ what sometimes gives ground to use such bi-quternions as a math tool in Special Relativity, the applications having rather formal character. More profound link with physics is discovered within pure vector-bi-quaternion numbers subset suggesting a natural version of relativity theory proved to easily describe arbitrary non-inertial (and of course inertial) motions of frames of reference.[11] Due to generic difficulties with definition of the norm the bi-quaternions are not represented as generalized Euler formula. There are few variants of simplest 2×2-matrix representations of a biquaternion in literature; one built on Pauli matrices $\mathbf{p}_k$: $\mathbf{q}_k = -i\mathbf{p}_k$

$$b = \begin{pmatrix} x - iy_1 & -y_3 - iy_2 \\ y_3 - iy_2 & x + iy_1 \end{pmatrix} \tag{3}$$

is cited more frequently. Section 2 contains study of generic properties of representations and interior structure of hypercomplex units. In Sections 3 split-complex and dual numbers are regarded as special cases of bi-quaternions. Short discussion is found in Section 4.

2. REPRESENTATIONS AND STRUCTURE OF HYPERCOMPLEX UNITS

Represent bi-quaternion (3) in the developed form

$$b = x + y_1 \mathbf{q}_1 + y_2 \mathbf{q}_2 + y_3 \mathbf{q}_3, \quad \{x,\ y_1,\ y_2,\ y_3\} \in \mathbf{C}. \tag{4}$$

A number of each of three good algebras can be selected from Eq. (4). If $\{x,\ y_1,\ y_2,\ y_3\} \in \mathbf{R}$, then number (4) is a quaternion. If $y_1 = y_2 = y_3 = 0$, and $x \in \mathbf{C}$ then Eq. (4) gives a complex number. If $y_1 = y_2 = y_3 = 0$, and $x \in \mathbf{R}$ then Eq. (4) gives a real number. This observation may seem trivial but only at a shallow glance. Write the coefficients at bi-quaternion (4) explicitly as complex numbers $x = r + i\,t$, $y_k = w_k + i z_k$(r, t, w_k, z_k are real numbers) and put the bi-quaternion in the form

$$b = r + w_1 \mathbf{q}_1 + w_2 \mathbf{q}_2 + w_3 \mathbf{q}_3 + i\,t + z_1 \mathbf{p}_1 + z_2 \mathbf{p}_2 + z_3 \mathbf{p}_3 \tag{5}$$

($\mathbf{p}_k$ are Pauli-type matrices); thus the hypercomplex number is shown built onto reciprocally imaginary pairs of four units.

Now consider an example, a special case of Eq. (5) $w_2 = w_3 = z_k = 0$, $w_1 \equiv w \neq 0$, $\mathbf{q}_1 \equiv \mathbf{q}$

$$b = r + w\mathbf{q} + i\,t \tag{6}$$

The answer onto the question "what type of a number is it?" is not straightforward, it strongly depends on the nature of the imaginary unit i.

2.1. Case A

If i is a scalar (as Hamilton primarily suggested[8]), then the number (6) can be generally classified as a cut bi-quaternion, the so called bi-complex number with $\mathbf{q}$ being an imaginary unit $\mathbf{q}^2 = -1$ of different (vector) nature than that of i. If unit $\mathbf{q}$ is represented by a 2×2-matrix (vector form), then $(r + i\,t)$ is a multiplier of a unit 2×2-matrix.

2.2. Case B

But it is well known that imaginary unit i itself can be represented as a vector-matrix $\mathbf{i}$, e.g.,

$$\mathbf{i} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{7}$$

In this case referring number (6) to a certain set depends on relations between $\mathbf{i}$ and $\mathbf{q}$. Let the units be given by a 2×2-matrix

generally with complex components. In this case three different types of number (6) are possible:
(i) b is usual complex number $b = r + \mathbf{i}(t + w)$ if $\mathbf{q}$ is defined by Eq. (7), i.e., $\mathbf{q} = \mathbf{i}$, in particular b is a real number if $t = -w$;
(ii) b is a quaternion if $\mathbf{q} \neq \mathbf{i}$ with $Tr(\mathbf{i}\cdot\mathbf{q}) = 0$, the last equality guarantees that the units $\{\mathbf{q},\ \mathbf{i}\}$ form compatible quaternion basis, this is e.g., realized when

$$\mathbf{q} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$$

(iii) b can not be referred to any definite set of numbers if $\mathbf{q} \neq \mathbf{i}$, and $Tr(\mathbf{i}\cdot\mathbf{q}) \neq 0$.

2.3. Case C

Both imaginary units may be given by 4×4-matrix with real components; e.g., if

$$\mathbf{q} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad \text{then} \quad \mathbf{i} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\text{restores the Case A, while} \quad \mathbf{i} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

corresponds to the Case B (ii).

Thus example of the number (6) shows that understanding of hypercomplex (HC) numbers (linked to understanding of fundamental physics) should be started by thorough study of the basic units forming associated algebras. Two main aspects of the study given below are representation and structure of the units.

2.4. Representation of the HC-Units

Operation with HC-numbers can be performed without any representation, the units given symbolically, e.g., $\{1,\ \mathbf{i},\ \mathbf{j},\ \mathbf{k}\}$, Hamilton notation, or $\{1,\ \mathbf{q}_k\}$, tensor notation. Nonetheless the units can be represented explicitly by square matrices with components comprising "simpler" numbers. Thus 2×2-matrix representations of the units comprise complex-number components, with the help of Eq. (7) the representations may be transformed into 4×4-matrices with only real components. Of course higher ranks of matrices-HC-units exist since the real unity may be regarded as a unit matrix of a certain rank, but this augmentation of the rank seems to be an artificial operation.

HC-representation in matrices of any rank is not unique. It is easily verified that the multiplication rule [2a] keeps its form if vector units $\mathbf{q}_k$ are subject to transformations $\mathbf{q}_{k'} = O_{k'n}\mathbf{q}_n$ from the group of special 3D-rotations over field of complex numbers, $O_{k'n} \in SO(3, C)$, or to transformations $\mathbf{q}_{k'} = U\mathbf{q}_k U^{-1}$ from the special linear ("reflection") group $U \in SL(2, C)$,[9] the groups being respectively 1:2 isomorphic to each other and similarly isomorphic to the Lorentz group. One notes that the transformations don't involve the real (scalar) unit, so form of the unit matrix remains intact. Thus each triad of 2×2-matrices

$$\mathbf{q}_1 = -i\begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \quad \mathbf{q}_2 = -i\begin{pmatrix} e & f \\ g & -e \end{pmatrix}, \quad \mathbf{q}_3 = \mathbf{q}_1\mathbf{q}_2 \tag{8}$$

form a vector HC-basis, a, b, c, e, f, g being complex numbers (or functions) satisfying the following restrictions $\det\mathbf{q}_1 = a^2 + bc = 1$, $\det\mathbf{q}_2 = e^2 + fg = 1$, $Tr\,\mathbf{q}_3 = 2ae + bg + cf = 0$.

Only three complex parameters are found independent providing 6 degrees of freedom characteristic for all mentioned above transformation groups. Typical representative of the group $SO(3, C)$ is a matrix of "simple rotation" (about vector $\mathbf{q}_3$),

$$O_{k'n} = \begin{pmatrix} \cos\Phi & \sin\Phi & 0 \\ -\sin\Phi & \cos\Phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Φ being a complex-number parameter; it is easily shown that an arbitrary matrix belonging to this group is reduced to product of three such simple rotations (in general case about each of three vector units). An elementary representative of the group $SL(2, C)$ performing similar rotation is a quaternion

$$U = \cos\frac{\Phi}{2} + \mathbf{q}_3\sin\frac{\Phi}{2}$$

together with its inverse

$$U^{-1} = \cos\frac{\Phi}{2} - \mathbf{q}_3\sin\frac{\Phi}{2}$$

As well an arbitrary transformation is reduced to subsequent implementation of transformations of this type (with different generating vectors).

Most primitive ("not transformed") vector HC-units are usually built upon by Pauli-matrices base

$$\mathbf{q}_{\tilde{1}} = -i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{q}_{\tilde{2}} = -i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\mathbf{q}_{\tilde{3}} = \mathbf{q}_{\tilde{1}}\mathbf{q}_{\tilde{2}} = -i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{9}$$

Subject to $SO(3, C)$-rotations or $SL(2, C)$-reflections the units (9) are transformed to a triad of the type (8). The latter can be put in the form with distinguished real and imaginary parts of components

$$\mathbf{q}_1 = -i\begin{pmatrix} \alpha + i\beta & \gamma + i\delta \\ \xi + i\zeta & -\alpha - i\beta \end{pmatrix}, \quad \mathbf{q}_2 = -i\begin{pmatrix} \eta + i\theta & \kappa + i\lambda \\ \mu + i\nu & -\eta - i\theta \end{pmatrix}$$

$$\mathbf{q}_3 = \mathbf{q}_1\mathbf{q}_2 \tag{10}$$

where α, β, γ, δ, ξ, ζ, θ, κ, λ, μ, ν are real numbers (functions), and 3 complex (6 real) equations to be fulfilled

$$(\alpha + i\beta)^2 + (\gamma + i\delta)(\xi + i\zeta) = 1 \tag{11a}$$

$$(\eta + i\theta)^2 + (\kappa + i\lambda)(\mu + i\nu) = 1 \tag{11b}$$

$$2(\alpha + i\beta)(\eta + i\theta) + (\gamma + i\delta)(\mu + i\nu) + (\xi + i\zeta)(\kappa + i\lambda) = 0 \tag{11c}$$

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Substitution of Eq. (7) unit 2×2-matrix into Eqs. (10) yields general form of 4×4-matrix representation of HC-units with only real components

$$\mathbf{q}_1 = -i\begin{pmatrix} \alpha & -\beta & \gamma & -\delta \\ \beta & \alpha & \delta & \gamma \\ \xi & -\zeta & -\alpha & \beta \\ \zeta & \varepsilon & -\beta & -\alpha \end{pmatrix}$$

$$\mathbf{q}_2 = -i\begin{pmatrix} \eta & -\theta & \kappa & -\lambda \\ \theta & \eta & \lambda & \kappa \\ \mu & -\nu & -\eta & \theta \\ \nu & \mu & -\theta & -\eta \end{pmatrix}$$

$$\mathbf{q}_3 = \mathbf{q}_1\mathbf{q}_2 \qquad (12)$$

the set of Eq. (11) holds. Further on only 2×2-matrices of the type (8) will be regarded.

2.5. Structure of the HC-Units

Thorough study shows that the units $\mathbf{q}_k$ are not most simple math objects but they are built of more elementary objects called and treated in literature differently. These objects with their properties definitely form a fundamental domain of algebra and geometry associated with HC-numbers, and deserves a separate profound study planned to be done. Only a sketch of the domain is given here, nonetheless demanding attention and space.

Basic notion will be an eigenfunction (EF) of any of the HC-unit $\mathbf{q}$ regarded as an operator. The operator given by a matrix from Eq. (8) may have left and right EF $\varphi\,\mathbf{q} = k\,\varphi$, $\mathbf{q}\psi = l\,\psi$, k, l being eigenvalues (EV). It is straightforwardly computed[9] that for any vector unit $\mathbf{q}_k$ all EV are $k^\pm = l^\pm = \pm i$, and for $\mathbf{q}$ equal e.g., to $\mathbf{q}_1$ from Eq. (8) explicit forms of left and right EF are respectively rows and columns

$$a\neq 1:\quad \varphi^\pm = \frac{1}{\sqrt{2}}\begin{pmatrix}\sqrt{1\mp a} & \mp\dfrac{b}{\sqrt{1\mp a}}\end{pmatrix}$$

$$\psi^\pm = \frac{1}{\sqrt{2}}\begin{pmatrix}\sqrt{1\mp a} \\ \mp\dfrac{c}{\sqrt{1\mp a}}\end{pmatrix} \qquad (13a)$$

$$a=1, b=0:\quad \varphi^+ = (-c/2 \quad 1),\quad \varphi^- = (1 \quad 0)$$

$$\psi^+ = \begin{pmatrix}0\\1\end{pmatrix},\quad \psi^- = \begin{pmatrix}1\\c/2\end{pmatrix} \qquad (13b)$$

$$a=1, c=0:\quad \varphi^+ = (0 \quad 1),\quad \varphi^- = (1 \quad b/2)$$

$$\psi^+ = \begin{pmatrix}-b/2\\1\end{pmatrix},\quad \psi^- = \begin{pmatrix}1\\0\end{pmatrix} \qquad (13c)$$

Similarly EF are found for an arbitrary vector HC-unit; altogether there are three couples of EF for each HC-triad $\{\varphi_k^\pm,\ \psi_k^\pm\}$: $\varphi_k^\pm\,\mathbf{q}_k = \pm i\,\varphi_k^\pm$, $\mathbf{q}_k\psi_k^\pm = \pm i\,\psi_k^\pm$ (no summation in k).

The EF belonging to any HC-unit have the following generic properties: (j) EF of same parity (+ or −) are normalized $\varphi^\pm\,\psi^\pm = 1$; (jj) EF of opposite parity are orthogonal $\varphi^\mp\,\psi^\pm = 0$; (jjj) tensor product of EF of same parity form an idempotent $\psi^\pm\,\varphi^\pm = C^\pm$, $\det C^\pm = 0$, $Tr\,C^\pm = 1$, $C^{\pm N} = C^\pm$, N is a natural number.

Due to these properties the following equalities hold $C^\pm\psi^\pm = \psi^\pm$, $\varphi^\pm C^\pm = \varphi^\pm$ i.e., EF of a HC-unit is as well an eigenvector of idempotent matrix C with eigenvalues equal to unity; in this case the EF are called ideals of idempotent.[10] Performing more profound analysis of the properties (j-jjj) one discovers that the idempotent (or tensor product of EF) is algebraically linked with its own HC-unit and real unit $C^\pm = (1\mp i\,\mathbf{q})/2$ what allows expressing the units through the EF:

$$1 = \psi^+\varphi^+ + \psi^-\varphi^- \qquad (14a)$$

$$\mathbf{q} = i\,(\psi^+\varphi^+ - \psi^-\varphi^-) \qquad (14b)$$

Now it is useful to introduce 2D matrix indices $A, B, D, E\ldots = 1, 2$, and denote $\varphi^+ \equiv h_{(1)A}$, $\psi^+ \equiv h^A_{(1)}$, $\varphi^- \equiv h_{(2)A}$, $\psi^- \equiv h^A_{(2)}$, free indices enumerate components of the matrices, lower and upper indices respectively count columns and rows. The units then are explicitly represented as matrices $1 = \delta^B_A$, $\mathbf{q} \equiv q^B_A$. This helps to understand geometric meaning of EF; indeed, with new notations Eq. (14a) and the normalization conditions acquire the form

$$\delta^B_A = h_{(D)A}h^B_{(D)} \qquad (15a)$$

$$h_{(D)A}h^A_{(E)} = \delta_{DE} \qquad (15b)$$

(summation rule holds). Eqs. (15) are nothing but famous correlations of differential geometry; they describe well known basic properties of Lame coefficients $h_{(D)A}$, in this case a dyad, linking at a point vicinity differentials of coordinates dy^A on a curved surface with those on its tangent plane $dX_{(D)}$: $dX_{(D)} = h_{(D)A}dy^A$. The metric of base surface $g_{AB} = h_{(D)A}h_{(D)B}$ and its inverse $g^{AB} = h^A_{(D)}h^B_{(D)}$ lowers and raise indices, e.g., $h_{(D)A} = g_{\mathrm{AB}}h^B_{(D)}$, their components are in general complex functions; Cartesian (Galilean) metric on a plane δ_{DE} has constant components and it is indifferent to the index position. Line element of the surface is defined as $dl^2 = \delta_{\mathrm{DE}}dX_{(D)}dX_{(E)} = g_{\mathrm{AB}}dy^Ady^B$. Leaving aside question of reality of the surfaces existence, just stress that EF, the most fundamental elements of HC-numbers, have definite geometric meaning as dyads.

Now use the dyad to rewrite Eq. (14b) $q^B_A = i\,(h_{(1)A}h^B_{(1)} - h_{(2)A}h^B_{(2)})$. Introducing 2D Minkowski-type metric

$$\eta_{DE} = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

one arrives at compact formula $q^B_A = i\,\eta_{DE}h_{(D)A}h^B_{(E)}$ which after lowering indices with the help of $g_{\mathrm{AB}} = h_{(D)A}h_{(D)B}$ gives

$$q_{\mathrm{AB}} = i\,\eta_{\mathrm{DE}}h_{(D)A}h_{(E)B} \qquad (16)$$

a symmetric in indices matrix object. Together with normalization condition (15b) Eq. (16) allows treating vector HC-unit q_{AB} as an indefinite metric of a specific "curve" 2D surface, then η_{DE} must be an image of q_{AB} onto imaginary tangent plane in vicinity of a mutual point. It seems helpful to emphasize that due to Eq. (6) any arbitrary vector HC-unit can be interpreted as such a metric.

As well a dyad components $h_{(1)A}$, $h_{(2)A}$ can be regarded as vectors on the base surface, hence matrices $C^+ \equiv C_{(1)AB} = g_{AB}$

$-h_{(2)A}h_{(2)B}$, $C^{-} \equiv C_{(2)AB} = g_{AB} - h_{(1)A}h_{(1)B}$ have math meaning of metrics on hyper-surfaces, orthogonal to subtracted vectors.

Another remarkable property of the EF as specific projectors is revealed in the study of ordinary (real) rotations of vector HC-units. Returning to initial notations remind that for a matrix-vector $\mathbf{a} \equiv a_k \mathbf{q}_k$ the contraction

$$\mp i\, \varphi_n^{\pm} \mathbf{a} \psi_n^{\pm} = \langle \mathbf{a} \rangle_n \tag{17}$$

gives value of the vector projection onto direction of vector $\mathbf{q}_n$ built by EF $\varphi_n^{\pm}$, $\psi_n^{\pm}$ (see e.g., Ref. [9]). It is worth noting that the form of Eq. (17) recalls procedures of finding values of quantum mechanical operators. This should not be an occasional coincidence since EF are in fact spinor functions similar to those entering quantum mechanical Schrödinger-Pauly equation and Dirac (more precisely, Weil) equations. Indeed, it is straightforwardly checked that transformation of a HC-unit by matrices from spinor group $SL(2, C)$ $\mathbf{q}' = U\mathbf{q}U^{-1}$ is equivalent to transformations of EF only $\psi'^{\pm} = U\psi^{\pm}$, $\varphi'^{\pm} = \varphi^{\pm}U$. Summing the above observation one concludes that most elementary (at present stage of investigation) math objects of HC-algebras are (a) eigenfunctions of vector HC-units, (b) ideals of idempotent matrices, (c) dyads (Lame coefficients) linking two surfaces, (d) projectors of matrix-vectors onto given axis, (e) spinor functions.

3. EXOTIC NUMBERS AS SPECIAL BI-QUATERNIONS

3.1. Split-Complex Numbers

If in the bi-quaternion (5) e.g., $w_k = t = z_2 = z_3 = 0$, $r \neq 0$, $z_1 \equiv z \neq 0$, (all real numbers), and $\mathbf{p}_1 \equiv \mathbf{p}$, then the HC-number is a split-complex number

$$b = r + z\,\mathbf{p} = r \cdot 1 + i\,z \cdot \mathbf{q} \tag{18}$$

Thus the "perplex" object is just a bi-quaternion with a real coefficient at unit 1 and with an imaginary coefficient at the only quaternion HC-unit $\mathbf{q}$, the latter multiplied by i (as a scalar) turns into Pauly-type unit $\mathbf{p}$. Eqs. (8), (10) and (12) demonstrate that there is infinite number of representations of the unit $\mathbf{q}$ (hence, $\mathbf{p}$). Treating numbers of the type (18) as a section of bi-quaternions helps to reveal math precipice between the involved units that seem to possess similar properties if the split-complex numbers are considered as a separate set. Indication onto bi-quaternion nature of the split-complex numbers exposes the difference: the real unit 1 obeys abelian multiplication law, while the unit $\mathbf{p}$ (or $\mathbf{q}$) is any of three HC-units composing a ring with non-abelian multiplication. Taking into account interior structure of the units one can represent the number (18) as 2×2-matrix in local basis $b_{AB} = (r\delta_{DE} + i\,z\,\eta_{DE})\, h_{(D)A} h_{(E)B}$, and in tangent basis $b_{(D)(E)} = r\delta_{DE} + i\,z\,\eta_{DE}$.

3.2. Dual Numbers

Consider bi-quaternion (5) with $w_2 = w_3 = t = z_1 = z_3 = 0$, $r \neq 0$, $w_1 = z_2 \equiv w \neq 0$, (real numbers)

$$b = r + w\,(\mathbf{q}_1 + \mathbf{p}_2) = r \cdot 1 + w \cdot (\mathbf{q}_1 + i\mathbf{q}_2) \tag{19}$$

Define the matrix $\varepsilon \equiv (\mathbf{q}_1 + i\mathbf{q}_2)/2$ and find that its square vanishes

$$4\varepsilon^2 = (\mathbf{q}_1 + i\mathbf{q}_2)(\mathbf{q}_1 + i\mathbf{q}_2) = \mathbf{q}_1^2 + i(\mathbf{q}_1\mathbf{q}_2 + \mathbf{q}_2\mathbf{q}_1) - \mathbf{q}_2^2 = 0$$

(factor 1/2 is introduced formally). This means that the cut bi-quaternion (19) is a dual number.

It is evident that any pair $\{\mathbf{q}_k,\, i\mathbf{q}_n\}$, $n \neq k$ of an arbitrary triad $\{\mathbf{q}_k\}$ form a dual number's "unit"

$$\varepsilon = (\pm\mathbf{q}_k \pm i\mathbf{q}_n)/2, \quad n \neq k \tag{20}$$

Using definition (20) and representation (10) one suggests a generic form of the matrix ε

$$\varepsilon = (\mathbf{q}_1 + i\mathbf{q}_2)/2 = -\frac{i}{2}\begin{pmatrix} \alpha - \theta + i\,(\beta + \eta) & \gamma - \lambda + i\,(\delta + \kappa) \\ \xi - \nu + i\,(\zeta + \mu) & -\alpha + \theta - i\,(\beta + \eta) \end{pmatrix}$$

implying that the ring conditions (11) for HC-triad hold. In particular matrix expressions for ε habitually used in literature are the following combinations of Eqs. (8)

$$\varepsilon^{\uparrow} = (i\,\mathbf{q}_1 - \mathbf{q}_2)/2 = \frac{1}{2}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\varepsilon^{\downarrow} = (i\,\mathbf{q}_1 + \mathbf{q}_2)/2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{21}$$

Matrices fit for dual numbers' nilpotent "unit" are also readily built from EF due to their fundamental property of orthogonality. Using EF of different parity but belonging to a certain HC-unit one constructs 2 such matrices for each HC-unit (the unit number not indicated) $\varepsilon^{\pm} = \psi^{\pm}\varphi^{\mp}$, or in dyad notations $\varepsilon_B^A = h_{(1)}^A h_{(2)B}$, $\varepsilon_B^A = h_{(2)}^A h_{(1)B}$; the square of the "unit" vanishes, e.g., : $(\varepsilon^{+})^2 = \psi^{+}\varphi^{-}\psi^{+}\varphi^{-} = 0$, or $\varepsilon_B^A\, \varepsilon_C^B = h_{(1)}^A h_{(2)B} h_{(1)}^B h_{(2)C} = h_{(2)}^A h_{(2)C}\delta_{12} = 0$. Compute for instance explicit form of the "unit" generated by matrix (7), [or $\mathbf{q}_{\tilde{2}}$ from Eqs. (9)]

$$\psi^{+} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \varphi^{-} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \end{pmatrix}; \quad \varepsilon^{+} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

the last matrix is expressed through HC-units $\varepsilon^{+} = -\mathbf{q}_{\tilde{2}} + i\,\mathbf{q}_{\tilde{3}}$ i.e., is satisfies general rule (20).

4. DISCUSSION

Summarizing the above analysis it is reasonable to stress that the "strange" split-complex and dual numbers habitually regarded as isolated and self-consistent sets could be in fact quite different sections of bi-quaternions comprising in their base apart of scalar unit various combinations of vector units with real or imaginary (from complex numbers algebra viewpoint) factors. Discussing fundamental and technical mathematical results one can state that the study of exotic numbers, at first, permits to reveal variability (non-uniqueness) of all types of the numbers' specialized units. Second, it is shown that any of HC-units of quaternion ring can become that of split-complex numbers but only when having imaginary coefficient. Otherwise, vector part of split-complex number can be treated as Pauly-type matrix with a real factor. As to dual numbers, their vector parts are algebraic sums of two different quaternion units (from same ring) with respectively real and imaginary coefficients equal in modulus. Touching physical aspects of the study one would emphasize comments on doubtful use of split-complex numbers as math base for 2D

"Lorentz physics" description and use of dual numbers as generator of 2D Galilean group. Such cut-space models seem to deviate from understanding real physical situations. Vice-versa, full quaternion numbers description of space-time appears much more adequate for the task; moreover the most natural implementation of quaternions offers though restricted but wider opportunities for construction successful theory of space-time and relative particles' motion.[11] But one has to recognize that the spinor structure of HC-units and associated geometry still remains a challenging domain of the fundamental math of HC-numbers closely linked with not less fundamental laws of physics. More profound study of the domain will be presented in following publications.

References and Notes

1. J. Cockle, *Philosophical Magazine* 33, 435 **(1848)**.
2. W. K. Clifford, Mathematical Works, edited by R. Tucker, Macmillan, London **(1882)**.
3. P. Fjelstad, *Am. J. Phys.* 54, 416 **(1986)**.
4. W. Band, *Am. J. Phys.* 56, 469 **(1988)**.
5. E. Study, Geometrie der Dynamen, B. G. Tuebner, Leipzig **(1903)**.
6. I. M. Yaglom, A Simple Non-Euclidean Geometry and Its Physical Basis, Springer Verlag, New York **(1979)**.
7. V. V. Kisil, Erlangen Program at Large–2: Inventing a Wheel. The Parabolic One, arXiv:0707.4024v1 [math.GM] July **(2007)**.
8. W. R. Hamilton, Lectures on Quaternions, Hodges and Smith, Dublin **(1853)**.
9. A. P. Yefremov, Quaternion and Biquaternions: Algebra, Geometry and Physical Theories, arXiv:math-ph/0501055v1, January **(2005)**.
10. P. Rastall, *Rev. Mod. Phys.* 2, 820 **(1964)**.
11. A. P. Yefremov, *Adv. Sci. Lett.* 1, 179 **(2008)**.

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