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Physical theories in hypercomplex geometric description

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Compact description is given of algebras of poly-numbers: quaternions, bi-quaternions, double (split-complex) and dual numbers. All units of these (and exceptional) algebras are shown to be represented by direct products of 2D vectors of a local basis defined on a fundamental surface. In this math medium a series of equalities identical or similar to known formulas of physical laws is discovered. In particular, a condition of the algebras' stability with respect to transformations of the 2D-basis turns out equivalent to the spinor (Schrödinger–Pauli and Hamilton–Jacobi) equations of mechanics. It is also demonstrated that isomorphism of $SO(3,1)$ and $SO(3,\mathbb{C})$ groups leads to formulation of a quaternion relativity theory predicting all effects of special relativity but simplifying solutions of relativistic problems in non-inertial frames. Finely it is shown that the Cauchy–Riemann type equations written for functions of quaternion variable repeat vacuum Maxwell equations of electrodynamics, while a quaternion space with non-metricity comprises main relations of Yang–Mills field theory.

Keywords: Hypercomplex numbers; quaternions; fundamental surface; spinors; quantum and classical mechanics; theory of relativity; electrodynamics; Yang–Mills field.

Mathematics Subject Classification 2010: 51P05

> "*I think it is practically certain that there is no chance whatever for Quaternions as a practical system of mathematics for the use of physicists. How is it possible, when it is so utterly discordant with physical notions, besides being at variance with common mathematics?*"[a]
>
> Oliver Heaviside

[a]Heaviside, O., "Electromagnetic theory", L.: The Electrician, Co. Vol. III, p. 519 (1912).

1. Introduction

The paper aims to offer in concentrated form results of the author's more than 30-year examination of the mathematics of hypercomplex numbers and its hidden links with formulations of physical laws. But it is necessary to emphasize that the idea of the paper is alien to attempts to just methodically rewrite well-known formulas in a new format; here we are committed to reveal and analyze objects and equalities immanently resident in the hypercomplex medium, but as well met in the precise or similar form of empiric and heuristic physical laws. The quaternion math is distinguished in this sense; these numbers constitute last in dimension associative but no more commutative division algebra. One discovers surprising similarity of some of its correlations with a series of physical laws. A possible reason for this is nearly incredible "geometricity" of quaternions. Thorough analysis shows that this mathematics not only naturally incorporates habitual features of 3D-world, but as well it seems to implicitly contain "more fundamental" pre-geometric structures though admitting visual images thus making easier comprehension of most abstract parts of physics, such as analytical and quantum mechanics. Moreover, the mathematical laws having clear geometrical sense give chance (of course with a certain degree of success) to introduce corrections into formulations of physical relations, and to construct respective models probably different from traditional ones.

The paper is organized as following. Section 2 is devoted to compact description of sets of poly-numbers and hypercomplex numbers. In particular, in Sec. 2.1 main relations of such associative algebras are given. In Sec. 2.2, a notion of fundamental pre-geometric surface is introduced with a local 2D-basis (dyad) on it fully determining all units of the associative algebras. Section 2.3 is devoted to description of 3D differential geometry on quaternion spaces. In Sec. 3, identity or similarity of the hypercomplex math relations with formulas describing physical laws are closely regarded. In Sec. 3.1, it is shown that the demand of the algebras' stability under transformations of constituting them dyad vectors is equivalent (dependently on the space-time scales) to equations of quantum or classical mechanics. In Sec. 3.2, on the base of isomorphism of the Lorentz group and the group preserving quaternion multiplication a vector version of relativity theory is formulated, and a series of relativistic problems is solved within its format. Section 3.3 demonstrates that relations characteristic for the gauge fields, electromagnetic field and Yang–Mills field, are discovered respectively in theory of function of quaternion variable, and in differential geometry of quaternion spaces. Brief discussion in Sec. 4 concludes the paper.

2. Math-Media of Hypercomplex Numbers

In this part, conceptions of hypercomplex numbers (and poly-numbers) are exposed together with versions of their representations and appropriate relations. When possible this data is related to geometric structures and objects associated with the texture of physical world.

2.1. *Hypercomplex numbers*

Real numbers are traditional (and justified) tool to describe results of physical experiments dealing with material objects and measurable magnitudes, while complex numbers are used at "intermediate" levels of computation in physical theories of twentieth century and actual today. Among them quantum mechanics, classical theory of fermion fields, quantum electrodynamics etc.; a trend appears to implement complex numbers even in classical theory of gravity.

Real and complex numbers have close algebraic properties, having though different number of units. Respective geometric images exist, a continuous infinite line for real numbers, and a surface, infinite or finite (Riemann sphere), for complex numbers. But it will be shown below that a closer look at the units' structure leads to a novel image of the complex number in the form of "conic gearing couple".

The term "hypercomplex" is attributed to numbers with more units than complex numbers. Normally this term is used for numbers constituting two "good" algebras, quaternions having four basic units, and octonions (Cayley algebra) having eight basic units. Sometimes the "hypercomplex set" is said to include also "poly-numbers", double (split-complex) numbers and dual numbers based on two different units, and bi-quaternions built on quaternion units but with complex-numbered coefficients.

Four "good" algebras of real numbers, complex numbers, quaternions and octonions are often called exceptional. The Frobenius–Gurwitz theorem proves that these four types of numbers constitute full set of finite-dimensional division algebras. But while algebra of quaternions (non-commutative ring) remains associative in multiplication, algebra of octonions is neither commutative, nor associative, the last property is replaced by a weakened version of "alternative multiplication" [1, 2]. No physical entities with non-associative multiplication are known, so octonions are not considered in this work; the other numbers deserve attention.

2.1.1. *Quaternion numbers and geometry*

A quaternion is an object $q = a1 + bi + cj + dk$ (Hamilton's notation) with real factors at one real (1) and three imaginary (i, j, k) units obeying the multiplication law

$$\begin{gathered} 1^2 = 1, \quad \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1, \quad 1\mathbf{i} = \mathbf{i}1 = \mathbf{i}, \quad 1\mathbf{j} = \mathbf{j}1 = \mathbf{j}, \quad 1\mathbf{k} = \mathbf{k}1 = \mathbf{k}, \\ \mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \quad \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \quad \mathbf{ki} = -\mathbf{ik} = \mathbf{j} \end{gathered} \tag{1}$$

(symbol "1" normally omitted); the multiplication table (1) includes 16 postulated equalities. Formulas are shorter in the vector notations $\mathbf{i}, \mathbf{j}, \mathbf{k} \to \mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3 \to \mathbf{q}_k$, $j, k, l, m, n, \ldots = 1, 2, 3$; then a quaternion written as a sum of scalar (a) and vector $(b_k\mathbf{q}_k)$ parts is $q \equiv a + b_k\mathbf{q}_k$, $a, b_k \in R$, and the law (1) has the form

$$1\mathbf{q}_k = \mathbf{q}_k 1 = \mathbf{q}_k, \quad \mathbf{q}_k\mathbf{q}_l = -\delta_{kl} + \varepsilon_{klj}\mathbf{q}_j, \tag{2}$$

summation in repeated indices is implied, δ_{kl}, ε_{jkl} are respectively 3D symbols of Kronecker and Levi-Civita (discriminant tensor).

Quaternions admit same operations as complex numbers. Comparison of quaternions is reduced to their equality (equality of coefficients at similar units). Commutative addition of quaternions is made by components. Quaternions are multiplied as polynomials by the rules (1) or (2); multiplication is non-commutative so left and right products are defined.

A quaternion $q = a + b_k \mathbf{q}_k$ has its conjugate $\bar{q} \equiv a - b_k \mathbf{q}_k$, the norm $|q|^2 \equiv q\bar{q} = \bar{q}q$ (a real number), and the modulus $|q| = \sqrt{q\bar{q}} = \sqrt{a^2 + b_k b_k}$. Inverse quaternion is $q^{-1} = \bar{q}/|q|^2$, so for two quaternions q_1 and q_2 division (left and right) is defined $(q_1/q_2)_{\text{right}} = q_1\bar{q}_2/|q_2|^2$, $(q_1/q_2)_{\text{left}} = \bar{q}_2 q_1/|q_2|^2$. If q is a product of two multipliers $q_1 = a + b_k \mathbf{q}_k$, $q_2 = c + d_k \mathbf{q}_k$ then definition of the norm yields

$$|q|^2 = |q_1 q_2|^2 = (q_1 q_2)\overline{(q_1 q_2)} = q_1 q_2 \bar{q}_2 \bar{q}_1 = q_1 \bar{q}_1 q_2 \bar{q}_2 = |q_1|^2 |q_2|^2, \tag{3a}$$

in the developed form Eq. (3a) is the identity of four squares

$$\begin{aligned}
&(ac - b_1 d_1 - b_2 d_2 - b_3 d_3)^2 + (ad_1 + cb_1 + b_2 d_3 - b_3 d_2)^2 \\
&\quad + (ad_2 + cb_2 + b_3 d_1 - b_1 d_3)^2 + (ad_3 + cb_3 + b_1 d_2 - b_2 d_1)^2 \\
&= (a^2 + b_1^2 + b_2^2 + b_3^2)(c^2 + d_1^2 + d_2^2 + d_3^2).
\end{aligned} \tag{3b}$$

"Identities of squares" exist only in four exceptional algebras: of real numbers (trivial identity), of complex numbers (identity of two squares), of quaternions Eq. (3), and of octonions (identity of eight squares).

Yet there is no satisfactory geometric image of all set of quaternions (like a plane for complex numbers) but these numbers are "very geometric". A quaternion with unit modulus may be uniquely related to an arc of a sphere's big circumference [3], while a product of two vector quaternions $\mathbf{a} = a_k \mathbf{q}_k$ and $\mathbf{b} = b_k \mathbf{q}_k$

$$\mathbf{ab} = a_k b_n \mathbf{q}_k \mathbf{q}_n = -a_k b_k + \varepsilon_{knm} a_k b_n \mathbf{q}_m,$$

comprises at once the Cartesian scalar product of vectors-multipliers and their vector product. This means that imaginary units behave as unit vectors initiating a Cartesian system of coordinates, the coefficients being the vectors' components. This observation first made by Hamilton pushed Heaviside and Gibbs to develop a simpler vector algebra.

The scalar unit here has no definite geometric image but it is distinct, so the number of the algebra's dimension (four) sometimes provokes 4D space-time associations, though non-fruitful physically because the real unit (allegedly time "direction") never changes. Instead another treatment of the scalar unit will be given in demonstration that the hypercomplex numbers have powerful "pre-geometric" foundation represented by a surface "underlying" the geometry of 3D space. The analysis given below strongly supports the idea of interior structure of physical world.

2.1.2. *"Bad" poly-number algebras*

Shortly expose three other associative algebras (having "disadvantages"), those of double numbers, dual numbers, and bi-quaternions.

Double numbers. (Split-complex numbers, hyperbolic numbers, tessarines, motors [4–6]) $s = x + jy$, where $x, y \in \mathbb{R}$, similar to complex numbers are built on two commuting units $1, j$, but square of each unit is a real unity $1^2 = 1, j^2 = 1$. These numbers admit addition, multiplication, and conjugation $s^* = x - jy$ same as complex numbers. The "norm" $\|s\|^2 = x^2 - y^2$ sometimes is claimed to represent a 2D Minkowski metric subject to respective Lorentz transformation [7]. If $\|s\| = 1$, then the Euler-type formula emerges $e^{j\eta} = \cosh\eta + j\sinh\eta$, so that $x = \cosh\eta$ where η is the hyperbolic parameter. But the number's "modulus" apparently can be an imaginary number or zero; this leads to defect of invertibility, hence of division. Geometrically a double number can be represented as a vector on a surface (plane). Two mutually conjugate vectors $e = (1+j)/2$, $e^* = (1-j)/2$ are orthogonal $ee^* = 0$ and idempotent $e^N = e, e^{*N} = e^*$ (N is a natural number); they form an orthogonal basis for any set of dual numbers $s = (x+y)e + (x-y)e^*$. The number $s = x + jy$ can be represented in the simplest matrix form $s = \begin{pmatrix} x & y \\ y & x \end{pmatrix}$, the matrix' determinant giving the modulus $\|s\| = \det s$. This matrix representation is not of course unique (see Sec. 2.2).

Dual numbers. (Parabolic numbers [8–10]) $d = x + \varepsilon y$, where $x, y \in \mathbb{R}$, are also built on two commuting units $1, \varepsilon$, the first unit is a real scalar $1^2 = 1$, the second is a nilpotent $\varepsilon^2 = 0$. Dual numbers admit addition, multiplication, and conjugation $d^* = x - \varepsilon y$ same as complex numbers. The norm of a dual number must be square of its real part $\|d\|^2 = d \cdot d^* = (x + \varepsilon y)(x - \varepsilon y) = x^2$, so the number's $d = \varepsilon y$ norm vanishes thus leading to the invertibility and division defects. But dual numbers possess a valuable property: the nilpotent unit ε can play role of an infinite parameter, so a function's full Taylor series is given by just two terms. For instance, a dual number with a unit norm ($x = 1$) is the full series of the exponent $e^{\varepsilon y} = 1 + \varepsilon y$ representing a dual analog of Euler formula. The exponent of the type $e^{\varepsilon t}$ (t is a parameter) behaves as an operator of a "parabolic rotation" of any dual number $d = x + \varepsilon y$, result of the rotation is a translation $e^{\varepsilon t} d = x + \varepsilon(y + xt)$ sometimes associated with the Galilean group. The simplest (not unique) matrix representations of dual numbers are $d^{\uparrow} = \begin{pmatrix} x & y \\ 0 & x \end{pmatrix}$, or $d^{\downarrow} = \begin{pmatrix} x & 0 \\ y & x \end{pmatrix}$.

Bi-quaternion numbers. [11–13] $b = x + y_k \mathbf{q}_k$, where $x, y_k \in \mathbb{C}$, the units $1, \mathbf{q}_k$ being those of quaternions, admit addition, multiplication, and conjugation $\bar{b} = x - y_k \mathbf{q}_k$ same as quaternions. But the norm is not well-defined since the product $b\bar{b} = x^2 + y_k y_k$ in general is not a real (positive) number. A real-number "norm" is appropriate for the subset of vector bi-quaternions $b = y_k \mathbf{q}_k$, $y_k = w_k + i z_k$ with orthogonal real and imaginary parts $w_k z_k = 0$, $\|b\|^2 = b\bar{b} = w_k w_k - z_k z_k$, there are evidently zero dividers, but it is this subset that comprises "best" formulas describing relative motion of frames of Ref. [14]. Simplest representation of biquaternions is given by the Pauli matrices $\mathbf{p}_k : \mathbf{q}_k = -i\mathbf{p}_k$

$$b = \begin{pmatrix} x - iy_1 & -y_3 - iy_2 \\ y_3 - iy_2 & x + iy_1 \end{pmatrix}.$$

It is straightforwardly shown that double and dual numbers are subsets of biquaternions [15].

2.2. *Pre-geometric basis of associative algebras*

2.2.1. *Units of associative algebras as functions of a dyad's vectors*

Let there exist a smooth 2D space (surface) endowed with a metric g_{AB} (and inverse: $g^{BC} \to g_{AB}g^{BC} = \delta_A^C$) and with a system of coordinates $x^A = \{x^1, x^2\}$; here $A, B, C, \ldots = 1, 2$, δ_A^C is 2D Kronecker symbol, summation in repeated indices is implied. A line element is

$$ds^2 = g_{AB}dx^A dx^B, \tag{4}$$

the surface may be curved so the metric covariant and contravariant components differ. In a point of the surface one can always choose a couple of unit orthogonal vectors a^A, b^B (a dyad)

$$g_{AB}a^A a^B = g_{AB}b^A b^B = 1, \tag{5}$$

$$g_{AB}a^A b^B = a^A b_A = 0. \tag{6}$$

A domain in vicinity of the dyad's origin (together with respective part of tangent plane having the metric $\delta_{MN} = \delta^{MN} = \delta_M^N$) will be called a "2D-cell".

Consider direct (tensor) products of the dyad vectors with mixed components [16]. One can construct only four such products (2×2-matrices), two idempotent matrices

$$G_B^A = a^A a_B, \quad H_B^A = b^A b_B \to G_B^A G_C^B = G_C^A, \quad H_B^A H_C^B = H_C^A \tag{7a}$$

and two nilpotent matrices

$$D_B^A = a^A b_B, \quad F_B^A = b^A a_B \to D_B^A D_C^B = 0, \quad F_B^A F_C^B = 0. \tag{7b}$$

Now build sum and difference of the idempotent matrices

$$E \equiv E_B^A \equiv G_B^A + H_B^A = a^A a_B + b^A b_B, \quad E^2 = E, \tag{8a}$$

$$\tilde{K} \equiv \tilde{K}_B^A \equiv G_B^A - H_B^A = a^A a_B - b^A b_B, \quad \tilde{K}^2 = E \tag{8b}$$

and sum and difference of the nilpotent matrices

$$\tilde{I} \equiv \tilde{I}_B^A \equiv D_B^A + F_B^A = a^A b_B + b^A a_B, \quad \tilde{I}^2 = E, \tag{8c}$$

$$J \equiv J_B^A \equiv D_B^A - F_B^A = a^A b_B - b^A a_B, \quad J^2 = -E. \tag{8d}$$

The set of objects (7) and (8) is sufficient to construct the basis of any algebra described above. The unit E is the basis of real numbers; the couple E and J form the basis of complex numbers; the units E and D (or E and F) form the basis of dual numbers; the units E and $\tilde{I}$ (or E and $\tilde{K}$) form the basis of double numbers. Finely if units (8b) and (8c) are "slightly corrected" so that their product is the third unit J, then one obtains the basis of quaternion (and bi-quaternion) numbers

$$1 \equiv E, \quad \mathbf{q}_1 \equiv -i\tilde{I}, \quad \mathbf{q}_2 \equiv J, \quad \mathbf{q}_3 \equiv i\tilde{K}. \tag{9}$$

Recall the spectral theorem (of the matrix theory) stating that any invertible matrix with distinct eigenvalues can be represented as a sum of idempotent projectors with the eigenvalues as coefficients, the projectors being direct products of vectors of a bi-orthogonal basis. The unit $\mathbf{q}_3$ defined in Eqs. (9) and (8b) is the characteristic example

$$\mathbf{q}_3|^A_B = ia^A a_B - ib^A b_B = iG^A_B - iH^A_B, \tag{10}$$

here right and left eigenfunctions of $\mathbf{q}_3$ are respectively vectors a^A, b^B and covectors a_A, b_B of the dyad, the eigenvalues are $+i$ (for a) and $-i$ (for b), and G^A_B, H^A_B are the projectors.

On the other hand, the similarity transformation of the units (9)

$$\mathbf{q}'_k \equiv \hat{S}\mathbf{q}_k\hat{S}^{-1}, \tag{11}$$

preserves the form of multiplication law (2), and if $\det \hat{S} = 1$, then $\hat{S} \in \mathrm{SL}(2, \mathbb{C})$, the spinor group performing generalized rotations (reflections) of the vectors. Therefore, vector units from Eq. (9) can be obtained from a single unit, say, $\mathbf{q}_3$ by a transformation (11), so that all vector units have same eigenvalues $\pm i$, and the eigenfunctions of the derived units are linear combinations of the eigenfunctions of the initial unit [17]. This also means that the mapping (11) is a secondary one, but the primary ones are $\mathrm{SL}(2, \mathbb{C})$-transformations of a dyad, the latter thus consisting of a set of spinors (from viewpoint of the 3D space described by the triad vectors $\mathbf{q}_k$).

Introduce a shorter 2D-index-free matrix notation for dyad spinor vector (a vector is a column, a covector is a row); a parity indicator $^+$ or $^-$ marks sign of the eigenvalue $\pm i$

$$a^A \to \psi^+, \quad a_A \to \varphi^+, \quad b^A \to \psi^-, \quad b_A \to \varphi^-. \tag{12}$$

Now the dyad orthonormality conditions (5), (6) have the form

$$\varphi^\pm\psi^\pm = 1, \quad \varphi^\mp\psi^\pm = \varphi^\pm\psi^\mp = 0, \tag{13}$$

the idempotent projectors are denoted as

$$C^+ \equiv G = \psi^+\varphi^+, \quad C^- \equiv H = \psi^-\varphi^-$$

and the matrix units (9) are

$$1 = \psi^+\varphi^+ + \psi^-\varphi^-, \tag{14a}$$

$$\mathbf{q}_1 = -i(\psi^+\varphi^- + \psi^-\varphi^-), \tag{14b}$$

$$\mathbf{q}_2 = \psi^+\varphi^- - \psi^-\varphi^-, \tag{14c}$$

$$\mathbf{q}_3 = i(\psi^+\varphi^+ - \psi^-\varphi^-). \tag{14d}$$

The transformation (11) clearly follows from the SL$(2,\mathbb{C})$-transformations of the spinor vectors

$$\psi'^{\pm} = S\psi^{\pm}, \quad \varphi'^{\pm} = \varphi^{\pm} S^{-1}. \tag{15}$$

Emphasize some primary results.

(1) Units of the considered above associative algebras are structured objects made up of "more elementary" spinor vectors of a dyad chosen on a 2D-cell of a fundamental surface; different 2D-cells provide different representations of the algebraic units.
(2) New: the scalar unit has sense of a metric of 2D-cell.
(3) If a triad $\mathbf{q}_k$ is an image of 3D space geometry, then the fundamental surface (and its 2D-cell) should be associated with an image of pre-geometry (the term suggested by Wheeler [18]), each pre-geometric dimension being a "square root" from a 3D (physical) dimension.
(4) The 3D space can be regarded as a multitude of small 3D bits ("cubes") each erected on its own 2D-cell, while the fundamental surface composed of many tightly gathered 2D-cells can be thought of as kind of a pre-geometric "world screen".

2.2.2. "*Conic gearing*" *image of a complex number*

Using Eqs. (14a), (14d) represent a complex number in the matrix form

$$z \equiv x + y\mathbf{q}_3 = x(\psi^+\varphi^+ + \psi^-\varphi^-) + iy(\psi^+\varphi^+ - \psi^-\varphi^-) = r\mathrm{e}^{i\beta}C^+ + r\mathrm{e}^{-i\beta}C^-, \tag{16a}$$

where $r = \sqrt{x^2+y^2}$, $\tan\beta = y/x$; imaginary unit here is $\mathbf{q} = \mathbf{q}_3$ but it may be any unit $\mathbf{q}_k$ each admitting expansion (14d). The number z in the form (16a) unites a scalar complex number and its conjugate, each component on a plane "oriented" by projectors C^+ and C^-, so the planes are orthogonal. Radius r cuts a disk in each plane; one disk rotated by angle β compels the second disk rotate by angle $-\beta$. This fits to the model of "conic gearing couple" [19, 20] (Fig. 1) built of two equal orthogonal gears touching each other in an edge-point rotating (without slipping) on orthogonal shafts; the projectors C^+, C^- are visualized as two fluxes parallel to the shafts. Somewhat different (but related) image of the complex number (16a) is given by just the two shafts of the conic couple, each having the length r and endowed with a flag (Penrose type flag [21]) indicating angle $\pm\beta$ between the radius and its initial position.

A rotation of the conic couple can be related to the dyad phase transformation (15)

$$\psi'^{\pm} = \mathrm{e}^{\pm i\alpha}\psi^{\pm}, \quad \varphi'^{\pm} = \mathrm{e}^{\mp i\alpha}\varphi^{\pm}, \tag{16b}$$

"exteriorly" preserving vector $\mathbf{q}_{3'} = \mathbf{q}_3$, two other vectors $\mathbf{q}_{1'}$, $\mathbf{q}_{2'}$ rotated about $\mathbf{q}_3$ by angle 2α. Equations (16b) imply that the dyad vectors may have real and

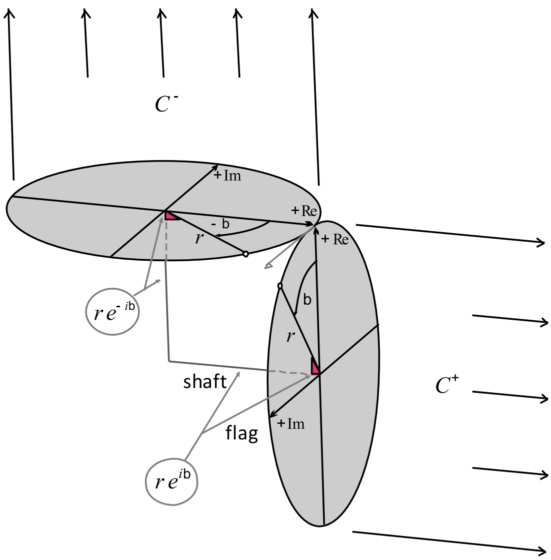

Fig. 1. Conic-gearing image of a complex number.

imaginary constituents

$$\psi'^{\pm} = (\cos\alpha \pm i\sin\alpha)\psi^{\pm}, \tag{17}$$

so each 2D-cell has a real and an imaginary areas; the changing phase makes one area to be "pumped over" into another. This "flickering" of 2D-cell is of course unobservable, its 3D manifestation is the triad $\mathbf{q}'_k$ rotation by a doubled angle. But pre-geometric peculiarities of a 2D-cell become "noticeable" mathematically when its dyad vectors are subject to a distorsion.

2.2.3. *Conformal stretching of a flickering* 2D-*cell and the algebras' stability*

Let the spinor vectors (16b) of a 2D-cell be subject to a conformal transformation

$$\psi''^{\pm} \equiv \sigma\psi' = \sigma e^{\pm i\alpha}\psi^{\pm}, \quad \varphi''^{\pm} \equiv \sigma\varphi' = \sigma e^{\mp i\alpha}\varphi^{\pm}, \tag{18}$$

the scale factor is a real number $\sigma \in \mathbb{R}$; $\sigma \neq 0$, $\sigma \neq 1$. Make a new notation

$$\lambda \equiv \sigma e^{i\alpha}, \tag{19}$$

then $\lambda^* = \sigma e^{-i\alpha}$, and $\psi''^{+} = \lambda\psi^{+}$, $\psi''^{-} = \lambda^*\psi^{-}$; $\varphi''^{+} = \lambda^*\varphi^{+}$, $\varphi''^{-} = \lambda\varphi^{-}$.

Vectors of new basis (18) are still orthogonal $\varphi''^{\mp}\psi''^{\pm} = 0$, but not unit

$$\varphi''^{\pm}\psi''^{\pm} = \lambda\lambda^* = \sigma^2, \tag{20}$$

i.e. a metric defect arises cancelling the option (14) to build algebras units from $\varphi''^{\pm}$, $\psi''^{\pm}$. However, a special condition can be introduced smoothing the defect away.

Let $\xi_\Lambda (\Lambda = 1, 2, \ldots, M)$ be coordinates in an abstract space $\boldsymbol{P}$, and θ a free parameter (θ, ξ_Λ "physically" unitless). Let also $\lambda(\theta, \xi_\Lambda)$ be a compact function in a volume V_Λ of $\boldsymbol{P}$ so that

$$f \equiv \int_{V_\Lambda} \lambda\lambda^* dV_\Lambda = 1. \tag{21}$$

Then the metric defect "is not seen" from $\boldsymbol{P}$, and the algebras' units can be built from $\varphi''^{\pm}$, $\psi''^{\pm}$

$$I'' = f(\psi''^{+}\varphi''^{+} + \psi''^{-}\varphi''^{-}) = I, \quad \mathbf{q}_{3''} = if(\psi''^{+}\varphi''^{+} - \psi''^{-}\varphi''^{-}) = \mathbf{q}_3, \tag{22a}$$

$$\mathbf{q}_{1''} = -if(\psi''^{+}\varphi''^{-} + \psi''^{-}\varphi''^{+}) = (\cos 2\alpha)\mathbf{q}_1 + (\sin 2\alpha)\mathbf{q}_2. \tag{22b}$$

However, the units (22) remain functions of θ; a condition of the units (or of the algebras) stability in the sense of the parameter can be imposed

$$\partial f/\partial\theta \equiv \partial_\theta f = 0. \tag{23}$$

Application of the Stokes theorem reduces Eq. (23) to the continuity-type equation

$$\partial_\theta(\lambda\lambda^*) + \nabla_\Lambda(\lambda\lambda^* k_\Lambda) = 0, \tag{24}$$

where k_Λ is a "2D-cell propagation vector" in the space, possibly given in various ways. It will be shown below that Eqs. (24) and (21) are most closely related to basic laws of physics.

2.3. *Geometry of quaternion spaces*

The Hamilton's discovery that the units $\mathbf{q}_k$ behave as a Cartesian frame was the first "physical law" identified in the quaternion math. Later, for about 100 years, examination of the frames and respective spaces was abandoned; however, the last decades demonstrate increase of interest to the subject [22]. So a brief representation of quaternion 3D geometric objects as a medium hiding more physical laws seems useful.

2.3.1. *Vector transformations of quaternion units*

The $\mathrm{SL}(2,\mathbb{C})$ transformations (11) keeping the law (2) form-invariant have $\mathrm{SO}(3,\mathbb{C})$ vectorial analog represented by orthogonal 3×3-matrices $O_{k'n}$ operating on the triads

$$\mathbf{q}_{k'} \equiv O_{k'n}\mathbf{q}_n. \tag{25}$$

If matrix parameters are real $O_{k'n} \in \mathrm{SO}(3,\mathbb{R})$, then Eq. (25) describes the triad's $\mathbf{q}_k$ space rotation; imaginary parameters imply hyperbolic rotations (though $\mathbf{q}_{k'}$ still satisfy the law (2)). A rotation about one unit vector, e.g. about $\mathbf{q}_3$ by angle γ, is a "simple rotation", its matrix denoted as $O_{k'n} \to O_3^\gamma$ (see e.g. Eq. (22), $\gamma = 2\alpha$); any matrix $O_{k'n}$ is a product of simple rotations.

Each vector-quaternion $\mathbf{a} \equiv a_k \mathbf{q}_k$ is form-invariant under $\mathrm{SO}(3,\mathbb{R})$ rotations

$$\mathbf{a} = a_k \mathbf{q}_k = a_{k'} \mathbf{q}_{k'}, \tag{26}$$

since $a_k = O_{n'k} a_{n'}$ and $O_{n'k} O_{n'm} = \delta_{km}$. For complex numbered $\mathrm{SO}(3,\mathbb{C})$-parameters the vector bi-quaternions of the type $\mathbf{z} = (a_k + ib_k)\mathbf{q}_k = \mathbf{a} + i\mathbf{b}$ are regarded in physics, their "norm" being

$$\mathbf{z}\bar{\mathbf{z}} = (a_n + ib_n)(a_n + ib_n) = a^2 - b^2. \tag{27}$$

Equation (27) demands that the vectors $\mathbf{a}$ and $\mathbf{b}$ be orthogonal

$$a_{k'} b_{k'} = a_n O_{nk'} b_m O_{mk'} = \delta_{mn} a_n b_m = 0,$$

if e.g. $\mathbf{q}_1$ is aligned with $\mathbf{b}$, then $\mathbf{z} = ib_1\mathbf{q}_1 + a_2\mathbf{q}_2 + a_3\mathbf{q}_3$. The bi-quaternion's form-invariance

$$z = ib_1\mathbf{q}_1 + a_2\mathbf{q}_2 + a_3\mathbf{q}_3 = ib_{1'}\mathbf{q}_{1'} + a_{2'}\mathbf{q}_{2'} + a_{3'}\mathbf{q}_{3'} \tag{28}$$

demands [23] that matrices $O_{k'n} \in \mathrm{SO}(3,\mathbb{C})$ be products of simple rotations performed in any order, but the space rotations about "imaginary axes" (here O_1^{γ}), hyperbolic rotations about "real axes" (here $O_2^{i\eta}$ or $O_3^{i\chi}$), the set forming a subgroup $\mathrm{SO}(1,2) \subset \mathrm{SO}(3,\mathbb{C})$. These objects will be shown to form the base of quaternion version of relativity theory.

2.3.2. *Differentiation of triads and quaternion spaces*

If a triad's vector $\mathbf{q}_k$ is a smooth function of parameters Φ_ξ then its differential is expressed through proper connection $\omega_{\xi kn}$ and vectors of the same triad [24] (summation in $\xi = 1, \ldots, G$)

$$d\mathbf{q}_k(\Phi) = \omega_{\xi kn} \mathbf{q}_n d\Phi_\xi,$$

the connection antisymmetric in vector indices has $3G$ components. The triad's derivative is

$$\frac{\partial}{\partial \Phi_\xi} \mathbf{q}_k(\Phi) \equiv \partial_\xi \mathbf{q}_k(\Phi) = \omega_{\xi kn} \mathbf{q}_n. \tag{29}$$

The proper connection $\omega_{\xi kn}$ can be computed e.g. as a function of $\mathrm{SO}(3,\mathbb{C})$ matrices

$$\omega_{\xi km} = \partial_\xi O_{k\tilde{n}} O_{m\tilde{n}}, \quad \mathbf{q}_k \equiv O_{k\tilde{n}} \mathbf{q}_{\tilde{n}}, \quad \mathbf{q}_{\tilde{n}} = \text{const}.$$

The transformational properties of $\omega_{\xi kn}$ are straightforwardly established

$$\mathbf{q}_k = O_{kp'} \mathbf{q}_{p'} \to \omega_{\xi kn} = O_{kp'} O_{mn'} \omega_{\xi p'n'} + O_{mp'} \partial_\xi O_{kp'}, \tag{30}$$

meaning that the connection is not a tensor.

Introduce a 3D vector quaternion space $\mathbf{U}_3$ such that each of its point marked in (holonomic) coordinates y^a is an origin of a triad $\mathbf{q}_k$, and the triad's orientation functions $\Phi_\xi(y^a)$ are given. In general, $\mathbf{U}_3$ can be curved but admitting in each its point a plane tangent space $T(\mathbf{U}_3)$ whose coordinates x_n are locally linked to y^a by Lame coefficients

$$dx_k = g_{ka}dy^a, \quad g_{ka}g_n^A = \delta_{kn}, \quad g_{ka}g_k^b = \delta_a^b.$$

Define in $T(\mathbf{U}_3)$ parallel transport of a vector B_n

$$d_pB_n = -\Omega_{jkn}B_kdx_j,$$

where $\Omega_{jkn} = \Phi_{jkn} + \omega_{jkn} + \sigma_{jkn}$ is an affine connection comprising the standard Ricci rotation coefficients $\Phi_{jkn} = g_n^a\nabla_j g_{ka}$, proper connection ω_{jkn}, and an arbitrary term σ_{jkn}. Respective covariant derivative is conventionally defined as $D_jB_n \equiv \partial_jB_n + B_k\Omega_{jkn}$. Quaternion triads are not covariantly constant with respect to the derivative $D_j\mathbf{q}_n = \partial_j\mathbf{q}_n + \mathbf{q}_k\Omega_{jkn} = \mathbf{q}_k(\Phi_{jkn} + \sigma_{jkn})$, so the sum $\hat{\sigma}_{jkn} \equiv \Phi_{jkn} + \sigma_{jkn}$ represents total quaternion non-metricity compelling the triad rotate in addition to rotation caused by its proper connection.

The standard analysis of the first Cartan's structure equation on $T(U_3)$ gives links between basic 1-forms, connection 1-forms, and torsion 2-forms of the space, while the second structure equation yields components of the curvature tensor

$$R_{kmij} = \partial_i\Omega_{jkm} - \partial_j\Omega_{ikm} + \Omega_{jkn}\Omega_{inm} - \Omega_{ikn}\Omega_{jnm}.$$

One finds that part of the tensor built of only proper connection ω_{jkn} identically vanishes, so nonzero part of the curvature $R_{kmij} = {}^{\Phi}R_{kmij} + {}^{Q}R_{kmij}$ contains the Ricci coefficients

$${}^{\Phi}R_{kmij} = \hat{D}_i\Phi_{jkm} - \hat{D}_j\Phi_{ikm} + \Phi_{jkn}\Phi_{inm} - \Phi_{ikn}\Phi_{jnm}$$

and pure quaternion non-metricity

$${}^{Q}R_{kmij} = \tilde{D}_i\sigma_{jkm} - \tilde{D}_j\sigma_{ikm} + \sigma_{jkn}\sigma_{inm} - \sigma_{ikn}\sigma_{jnm} \tag{31}$$

(covariant derivatives in Eq. (31) with respect to the proper connection only).

The given above math information is sufficient to represent specific hypercomplex media where a band of physical laws dwells; they are examined in the next section of this review.

3. Physical Laws in Hypercomplex Mathematics

It is demonstrated below that some equalities natural for mathematics of hypercomplex numbers have format of physical laws or relations known from empirical or heuristic considerations. Among these inequalities we shall meet equations of quantum and classical mechanics, theory of relativity, equations of electrodynamics, and those of Yang–Mills field.

3.1. *Spinor equations of mechanics*

This section is based on the introduced above notion of a 2D-cell of the fundamental pre-geometric surface "underlying" the 3D world.

3.1.1. *Equation of quantum mechanics* (*Schrödinger equation*)

The theory of quantum mechanics is based on the equation heuristically suggested by Schrödinger in 1926; it fits well to results of experiments and now is considered conventional. One easily finds that this equation is a special case of the algebras' stability condition (24)

$$\lambda^* \partial_\theta \lambda + \lambda \partial_\theta \lambda^* + \partial_\Lambda (\lambda \lambda^* k_\Lambda) = 0. \tag{32}$$

Let vector k_Λ indicate direction of the phase increase $k_\Lambda = \partial_\Lambda \alpha$, the phase function expressed from $\lambda = \sigma e^{i\alpha}$ as $\alpha = \frac{i}{2} \ln \frac{\lambda^*}{\lambda}$. Then $k_\Lambda = \frac{i}{2}(\frac{\partial_\Lambda \lambda^*}{\lambda^*} - \frac{\partial_\Lambda \lambda}{\lambda})$; insert this into Eq. (32) and find after a simple algebra (with summation in $\Lambda = 1, \ldots, M$)

$$\partial_\theta \lambda - \frac{i}{2} \partial_\Lambda \partial_\Lambda \lambda + iW\lambda + e^{2\alpha} \left(\partial_\theta \lambda^* + \frac{i}{2} \partial_\Lambda \partial_\Lambda \lambda^* - iW\lambda^* \right) = 0,$$

where a free term $i\lambda\lambda^* W$ is added and subtracted. If this equation holds for all values of the phase then each of its conjugate parts vanish; equation for the function $\lambda(\xi_\Lambda, \theta)$ is

$$\left(\partial_\theta - \frac{i}{2} \partial_\Lambda \partial_\Lambda + iW \right) \lambda = 0. \tag{33}$$

If the abstract math space $\boldsymbol{P}$ is the 3D physical world then the unitless coordinates become the space coordinates measured in units of length and the free parameter becomes time

$$\xi_\Lambda \to x_k/\varepsilon, \quad \theta \to t/\tau. \tag{34}$$

Choose the scale standards e.g. as

$$\varepsilon \equiv \frac{\hbar}{mu}, \quad [\varepsilon] = \mathrm{cm}; \quad \tau \equiv \frac{\varepsilon}{u} = \frac{\hbar}{mu^2}, \quad [\tau] = s, \tag{35}$$

where $\hbar$ is the Planck constant, m is a characteristic mass, u is a certain velocity ($u_{\max} \to c$); then Eq. (33), with the substitutions (34), (35), becomes exactly the Schrödinger equation

$$\left(i\hbar \partial_t + \frac{\hbar^2}{2m} \partial_k \partial_k - U \right) \lambda(x,t) = 0, \tag{36}$$

the free function $U = mu^2 W$ acquiring sense of a potential. The functional (21) in physical space

$$\int_{V_\Lambda} \lambda \lambda^* dV_\Lambda \to \frac{1}{\varepsilon^3} \int_V \sigma^2 dV = 1, \tag{37}$$

offers an original geometric and physical treatment of the "wave function" λ. "Pre-geometrically" $\lambda(x,t) = \sigma(x,t) e^{i\alpha(x,t)}$ describes a σ-times stretched 2D-cell flickering in the complex plane with the phase α; the parameters now depend on the space points and time. Function σ can be regarded as a "relative semi-density" of mass $\sigma = \sqrt{\rho(x,t)/\rho_{\text{mean}}}$, where $\rho(x,t)$ is a local density, $\rho_{\text{mean}} = \text{const.}$ is a mean density over a 3D space volume V. Then Eq. (37) is definition of the mass

$$\int_V \rho(x,t)dV = \rho_{\text{mean}}\varepsilon^3 = m. \tag{38}$$

From the large scale viewpoint, this is a compact mass (a material point) with frozen-in triad rotated by angle 2α. This model helps to purposefully search for solutions of Eq. (36).

3.1.2. *Hydrogen atom: Schrödinger solution and Bohr model*

The Schrödinger model of hydrogen-type atom (widely considered the single one correct) emerges as an exact solution of Eq. (36) with the potential $U \equiv -q^2/r$ of central electric charge. Only the phase depends on time in the model $\lambda(r,t) \equiv \sigma(x)e^{-i\frac{Et}{\hbar}}$, $E = \text{const.}$, so it is classified as a stationary one. Here (for simplicity) only circle "orbitals" are regarded, i.e. the scale factor is a function of radius $\sigma = \sigma(r)$, then Eq. (36) in spherical coordinates has the form

$$E + \frac{\hbar^2}{2m}\frac{1}{r\sigma}\frac{d^2}{dr^2}(r\sigma) + \frac{q^2}{r} = 0. \tag{39}$$

Solutions of Eq. (39) (in the variables r/a, E/E_0, where $a = \hbar^2/(mq^2)$, $E_0 = mq^4/(2\hbar^2)$) are the Laguerre polynomials for the function σ/r and the energy levels $E = -E_0/n^2$.

"Pre-geometric" image of this model is a flickering (with frequency $\omega_n = E_n/\hbar$) 2D-cell compactly stretched in the neighborhood of each discrete radial level. "Geometrically" (in the physical space), this is a mass with a triad's origin in its center of symmetry, the triad rotating (with frequency $2\omega_n$) about an arbitrarily pointed vector $\mathbf{q}_3$. The mass is immobile in the space so the Schrödinger model is rather a static than a stationary one.

However, Eq. (36) with the same potential admits another solution describing steady motion of a mass (indeed stationary). For an orbiting electron one expects a wave solution with a phase

$$\alpha = k\varphi - \omega t, \tag{40}$$

where $k = \text{const.}$ is a wave number, φ is an azimuth angle of 3D spherical coordinates, $\omega = \text{const.}$ is a frequency of 2D-cell's harmonic flickering. The scale factor can also depend on time and coordinates (angular and radial), e.g. in the following combination of functions

$$\sigma = g(\varphi,t)f(r).$$

Then for circular orbits at polar angle $\pi/2$ Eq. (36) decays into the real and imaginary parts

$$\omega\hbar - \frac{k^2\hbar^2}{2mr^2} + \frac{q^2}{r} + \frac{\hbar^2}{2m}\left[\frac{1}{rf}\partial_r\partial_r(rf) + \frac{1}{gr^2}\partial_\varphi\partial_\varphi g\right] = 0, \tag{41a}$$

$$\partial_t g + \frac{k\hbar}{mr^2}\partial_\varphi g = 0. \tag{41b}$$

Variables in Eq. (41a) separate, the azimuthal equation $\partial_\varphi\partial_\varphi g + \mu^2 g = 0$ (μ is a real parameter) having the solution $g = \sin(\mu\varphi + \gamma)$. The normalization condition $\int_0^{2\pi} g^2 d\varphi = 1$ determines the amplitude value $A = 1/\sqrt{\pi}$ and imposes restrictions onto the parameter $\mu = n/2$, $n = 1, 2, 3, \ldots$.

The angular velocity of the orbiting electron $\Omega \equiv \dot{\varphi} = 2\dot{\gamma}/n$ can be also expressed through the 2D-cell flickering frequency $\Omega = 2\omega/k$, the two expressions compatible if $\dot{\gamma} = \omega$, $k = n$. So the azimuthal function is $g(\varphi, t) = \frac{1}{\sqrt{\pi}}\sin[\frac{n}{2}(\varphi - \Omega t)]$; inserted in Eq. (41b) it yields the expression

$$m\Omega_n r_n^2 = n\hbar, \tag{42}$$

well-known as formula of quantization of electron's angular momentum postulated by Bohr in his model of the hydrogen atom; here this formula appears as an exact solution of the Schrödinger equation. The discrete energy $E_n = \omega_n\hbar$, and momentum $p_n \equiv m\Omega r_n = n\hbar/r_n$ of an electron prompt to put Eq. (41a) into the different-scale form

$$E_n - \frac{p_n^2}{2m} + \frac{q^2}{r_n} = 0 = -\frac{\hbar^2}{2m}\left[\frac{1}{rf}\partial_r\partial_r(rf) + \frac{n^2}{4r^2}\right]. \tag{43}$$

Left-hand side of (43) describes electron's uniform circular motion in the central Coulomb force, the equal Newton's equation $p_n^2/(mr_n) = q^2/r_n^2$ giving the orbit's radius $r_n = an^2 = \hbar^2 n^2/(mq^2)$, the angular velocity of revolving $\Omega_n = mq^4/(n^3\hbar^3)$, the cyclic frequency of the 2D-cell's flickering $\omega_n = E_n/\hbar = \Omega_n/n$, i.e. all magnitudes of the Bohr's H-atom model. Right-hand side of (43) gives the amplitude $f_n(r)$, its knowledge though being formal since the model does not imply radial distribution of mass, the radius of each orbit fixed.

The picture of this solution drastically differs from that of the Schrödinger's one. "Pre-geometric" image of the Bohr-type model is a flickering (with frequency ω_n) and orbiting 2D-cell (propagating along the orbit as a wave with angular velocity Ω_n); the 2D-cell is stretched by factor σ_n. Before normalization the solution describes a harmonic distribution of the relative mass density $\sigma_n^2(\varphi, t)$ along the orbit's line, in fact a thin rotating ring with the number of density maximums equal to the orbit's number. After normalization (integration over the cyclic coordinate) the Bohr's point-mass electron is re-established, but now it carries a triad rotating with the frequency $2\omega_n$. The electron's orbital velocities are $u_n = \Omega_n r_n = u_1/n$, the basic velocity (at the 1st level) being $u_1 = \Omega_1 r_1 \cong c\tilde{\alpha}$ suggested by Sommerfeld [25], $\tilde{\alpha} \cong 1/137{,}036$ is the fine structure constant. The characteristic length in this case is radius of the first orbit $\varepsilon = \hbar/(mu) = r_1$.

3.1.3. *Spinor equations of mechanics*

Return back to Eq. (33) and recalling the shape of the function $\lambda = \sigma(\xi_\Lambda, \theta)e^{i\alpha(\xi_\Lambda, \theta)}$ separate the stability condition into real and imaginary parts (Bohm-type equations [26])

$$\partial_\theta \sigma + \partial_\Lambda \sigma \partial_\Lambda \alpha + \frac{1}{2}\sigma \partial_\Lambda \partial_\Lambda \alpha = 0, \tag{44a}$$

$$\partial_\theta \alpha + \frac{1}{2}\left(\partial_\Lambda \alpha\right)\left(\partial_\Lambda \alpha\right) + W - \frac{1}{2}\partial_\Lambda \partial_\Lambda \sigma / \sigma = 0, \tag{44b}$$

interpret each equation of the system. Equation (44a) is a continuity equation for conformal factor σ, multiplied by 2σ it becomes the continuity equation for density σ^2

$$2\sigma \partial_\theta \sigma + 2\sigma \partial_\Lambda \sigma \partial_\Lambda \alpha + \sigma^2 \partial_\Lambda \partial_\Lambda \alpha = \partial_\theta \sigma^2 + \partial_\Lambda(\sigma^2 \partial_\Lambda \alpha) = 0.$$

Interpretation of Eq. (44) depends on the chosen scale. If its functions change quickly then in physical units it is a part of Schrödinger equation just as (in special case) Eq. (39). But the phase α may change slowly compared to the factor σ; then the free function W in Eq. (44b) can be a sum $W = W_{\text{ext}} + W_{\text{int}}$ of "slow" (macro, exterior) part W_{ext} and of "fast" (micro, interior) part W_{int}. In this case, Eq. (44b) decays into two independent parts

$$\partial_\theta \alpha + \frac{1}{2}(\partial_\Lambda \alpha)(\partial_\Lambda \alpha) + W_{\text{ext}} = 0, \tag{45a}$$

$$\partial_\Lambda \partial_\Lambda \sigma / \sigma = 2W_{\text{int}}. \tag{45b}$$

First, analyze Eq. (45a); it evidently resembles the Hamilton–Jacobi equation of analytical mechanics, the phase α playing role of the action function. Express partial derivative through full derivative $\partial_\theta \alpha = d_\theta \alpha - (d_\theta \xi_\Lambda)(\partial_\Lambda \alpha)$ and compute the phase value on a segment $[\theta_1, \theta_2]$

$$\alpha = \int_{\theta_1}^{\theta_2} \left[d_\theta \xi_\Lambda \partial_\Lambda \alpha - \frac{1}{2}(\partial_\Lambda \alpha)(\partial_\Lambda \alpha) - W_{\text{ext}} \right] d\theta \equiv \int_{\theta_1}^{\theta_2} L(d_\theta \xi_\Lambda, \xi_\Lambda) d\theta, \tag{46}$$

the integrand in Eq. (46) is a Lagrangian of "math classical mechanics". Minimal phase value on the segment found as vanishing variation $\delta\alpha = 0$ entails the "math equation of dynamics"

$$d_\theta \left[\partial_{\text{K}} \alpha + \frac{\partial(\partial_\Lambda \alpha)}{\partial(d_\theta \xi_{\text{K}})}\left(d_\theta \xi_\Lambda - \partial_\Lambda \alpha\right) \right] + \partial_{\text{K}} W_{\text{ext}} = 0, \tag{47}$$

here $\partial_{\text{K}} \alpha \equiv P_{\text{K}}$ is "momentum", $d_\theta \xi_{\text{K}} \equiv V_{\text{K}}$ is "velocity", $-\partial_{\text{K}} W \equiv F_{\text{K}}$ is "exterior force". In its turn Eq. (45b) determines the factor σ influenced by some interior agents represented by the "potential" W_{int} at this stage of analysis chosen arbitrarily.

Now define a function $S \equiv \hbar\alpha$ and rewrite Eqs. (44a), (45) over space-time in physical units (34), (35). Then Eq. (44a) (if $\sigma = \sqrt{\rho(x,t)/\rho_{\text{mean}}}$, $V_m \equiv \partial_m S/m$) is reduced to the mass conservation law

$$\partial_t \rho + \partial_m(\rho V_m) = 0, \tag{48a}$$

Eq. (45a) becomes the classical Hamilton–Jacobi equation for a massive point

$$\partial_t S + \frac{1}{2m}(\partial_m S)(\partial_m S) + U_{\text{ext}} = 0, \tag{48b}$$

while Eq. (45b) regulates distribution of the mass semi-density in a small domain

$$\partial_m \partial_m \sigma - R_{\text{int}}\sigma = 0, \tag{48c}$$

where $R_{\text{int}} \equiv (2m/\hbar^2)U_{\text{int}}$ is a free function measured in cm^{-2} (as e.g. curvature tensor). In physical units the integrand of functional (46) takes the form of mechanical Lagrangian, and Eq. (47) becomes precisely the equation of Newton's dynamics.

3.1.4. *Free particle and De Broglie wave*

If 3D particle is a compact mass with frozen-in vector triad rotating e.g. about $\mathbf{q}_3$ so that on a trajectory segment dz the particle's velocity is $\mathbf{V} = V\mathbf{q}_3$, then on the mass-density border (of radius $\varepsilon/2$) a triad vector orthogonal to $\mathbf{q}_3$ (e.g. $\mathbf{q}_1$) depicts a cylindric helix with the line element

$$dl^2 = dz^2 + (\varepsilon/2)^2 d(2\alpha)^2 = dz^2 + \varepsilon^2 d\alpha^2. \tag{49}$$

Equation (49) yields the change of respective 2D-cell's flickering phase

$$d\alpha = \pm\frac{1}{\varepsilon}\sqrt{dl^2 - dz^2}. \tag{50}$$

Demand that linear velocity of $\mathbf{q}_1$ on the border (scale velocity) be maximal $u = c$, i.e. $\varepsilon = \hbar/(mc)$, then $dl = c\,dt$; in its turn $dz = V\,dt$; insert these expressions into Eq. (50) to achieve

$$d\alpha = \pm\frac{mc^2 dt}{\hbar}\sqrt{1 - \frac{V^2}{c^2}}, \tag{51}$$

the signs $\pm$ specifying right or left helicity. The phase value (in units of the Planck's constant) on a segment $[t_1, t_2]$ is found by integration of Eq. (51). The integral taken with the sign "minus"

$$\alpha\hbar = -mc^2\int_{t_1}^{t_2} dt\sqrt{1 - \frac{V^2}{c^2}} \equiv -mc\int_{t_1}^{t_2} ds \equiv S \tag{52}$$

has the precise form of the action for a free relativistic particle (with left helicity).

Consider a non-relativistic particle, $V \ll c$, then from Eq. (51) one finds

$$dS \cong \hbar d\alpha = -mc^2 dt + \frac{mV^2}{2} dt. \tag{53}$$

Here develop the full derivative, $d\alpha = \omega dt + k_n dx_n$ ($\omega \sim$ const. (negative), $k_j \to k_z \sim$ const.), add and subtract the term $(mV^2/2)dt$, and replace $Vdt = dz$, then Eq. (53) takes the form

$$dS = \omega\hbar dt + k_z \hbar dz = -\left(mc^2 + \frac{mV^2}{2}\right) dt + mV dz \equiv -E dt + p_z dz,$$

giving at once the particle's classical and quantum energy and momentum

$$\frac{\partial S}{\partial t} = -E = \omega\hbar, \quad \frac{\partial S}{\partial z} = p_z = k_z \hbar \to \frac{\partial S}{\partial x_n} = p_n = k_n \hbar. \tag{54a}$$

So the phase is $\alpha \sim (p_n x_n - Et)/\hbar$, and the particle's 2D-cell image is the De Broglie wave

$$\lambda(x_n, t) \sim \sigma \exp[i(p_n x_n - Et)/\hbar]. \tag{54b}$$

Thus Eqs. (54), previously heuristic quantum assumptions, here result from the particle's "helix model", which in its turn appears on pure math grounds; main points are listed below.

(1) The only assumption (that the bordering velocity of the triad rotation is that of light) leads to Lagrangian of a free relativistic particle, and confirms that the 2D-cell flickering phase is the mechanical action measured in units of the Planck's constant $\alpha = S/\hbar$.
(2) As well the phase α is proportional to a space-time line element "seen" in 3D space as an arc of circumference depicted by a "transverse" triad's vector in the particle's frame.
(3) For an immobile particle $d\alpha = \omega dt$, so $\omega\hbar = mc^2$; hence the 2D-cell's flickering (or 3D particle's rotation) must be permanent; for a moving particle $\omega\hbar = mc^2\sqrt{1 - V^2/c^2} \equiv m_V c^2$.
(4) The 2D-cell's function $\lambda(x_n, t)$ of a 3D free non-relativistic particle is the De Broglie's wave.

3.1.5. *Derivation of Pauli equation*

Equation (33) derived for the scalar λ is the simplest version of the stability condition. But the dyad vectors are spinors (18), not scalars. Consider even spinors (indicator + is omitted) $\psi'' = \lambda\psi$, $\varphi'' = \lambda^*\varphi$; $\psi, \varphi =$ const. Then the integral (21) has the form

$$f(\theta) \equiv \int_{V_n} \varphi'' \psi'' dV_n = 1 \tag{55}$$

(here $\lambda(\xi_n, \theta)$ is defined in 3D space). The spinor nature comes to play if a vector field $A_k(x_n, t)$ is presently able to affect the 2D-cell's propagation, e.g. as

$$k_n = \partial_n \alpha + A_n. \tag{56a}$$

In this case, the local 3D metric must be given in the Clifford formulation

$$\delta_{kn} \equiv \frac{1}{2}(\mathbf{p}_k \mathbf{p}_n + \mathbf{p}_n \mathbf{p}_k) \tag{56b}$$

(here Pauli-type matrices $\mathbf{p}_k \equiv i\mathbf{q}_k$ are convenient to use). Equations (56) are taken into account, the condition of the integral (55) conservation takes the form

$$\partial_\theta(\varphi\lambda^*\lambda\psi) + \frac{1}{2}(\mathbf{p}_m\mathbf{p}_n + \mathbf{p}_n\mathbf{p}_m)\partial_m[\varphi\lambda^*\lambda\psi(\partial_n\alpha + A_n)] = 0. \tag{57}$$

After differentiations Eq. (57) becomes

$$\begin{aligned}(\lambda^*\varphi)&\left[\left(\partial_\theta - \frac{i}{2}\partial_k\partial_k + \frac{1}{2}\partial_k A_k + A_k\partial_k + \frac{i}{2}A_kA_k + \frac{i}{2}\varepsilon_{kmj}\mathbf{p}_j\partial_m A_k + iW\right)(\lambda\psi)\right] \\ &+ \left[\left(\partial_\theta + \frac{i}{2}\partial_k\partial_k + \frac{1}{2}\partial_k A_k + A_k\partial_k - \frac{i}{2}A_kA_k + \frac{i}{2}\varepsilon_{mkj}\mathbf{p}_j\partial_m A_k - iW\right)(\lambda^*\varphi)\right] \\ &\times(\lambda\psi) = 0,\end{aligned}$$

where the free term $i(\lambda^*\varphi)\times(\frac{1}{2}A_kA_k + W)\times(\lambda\psi)$ is added and subtracted. The last equation disintegrates into two conjugate equations. The equation for the spinor vector is

$$\left[i\partial_\theta - \frac{1}{2}(-i\partial_k + A_k)(-i\partial_k + A_k) - \frac{1}{2}\mathbf{p}_k B_k - W\right]\psi'' = 0, \tag{58}$$

where $B_k \equiv \varepsilon_{kmn}\partial_m A_n$. Hermitian conjugation of Eq. (58) gives similar equation for the spinor covector. Transition to physical variables (34), (35) converts Eq. (58) into the known Pauli equation

$$\left[i\hbar\partial_t - \frac{1}{2m}\left(-i\hbar\partial_k + \frac{q}{c}\tilde{A}_k\right)\left(-i\hbar\partial_k + \frac{q}{c}\tilde{A}_k\right) - \frac{q\hbar}{2mc}\mathbf{p}_k\tilde{B}_k - U\right]\Psi = 0, \tag{59}$$

where q is the electric charge, $\tilde{A}_k \equiv \frac{mc^2}{q}A_k$, $\tilde{B}_k \equiv \frac{mc^2}{q}B_k$ are potential and intensity of the magnetic field, and $U \equiv mc^2W$ is a scalar potential. Thus the Pauli equation similar to that of Schrödinger's can be derived just mathematically. One may recall that the heuristic Pauli term $\frac{q\hbar}{2mc}\mathbf{p}_k\tilde{B}_k$ was once theoretically deduced in assumption that the electric charge interacts with micro-structure of a quaternion space [27].

3.2. *Quaternion theory of relativity*

The multiplication law (2) for quaternion (and bi-quaternion) units is form-invariant under $SO(3, \mathbb{C})$ transformations of the units (see Eq. (25)), while bi-quaternion vectors with definable "norm" (27) are form-invariant under

SO(1, 2) ⊂ SO(3, ℂ) transformations (see Eq. (28)). These facts together with known isomorphism between SO(3, ℂ) and the Lorentz group SO(1, 3) allow formulation of the quaternion version of the relativity theory.

3.2.1. *Vector interval, rotational equation, and effects of relativistic motion*

Any quaternion triad $\mathbf{q}_k$ is treated here as an observer's frame of reference Σ. Consider a form-invariant "physical" vector bi-quaternion with definable "norm"

$$ds \equiv idt\mathbf{q}_1 + dr\mathbf{q}_2 = idt'\mathbf{q}_{1'} + dr'\mathbf{q}_{2'}, \tag{60a}$$

where dr, dt being differentials of a particle's coordinate and time observed from the frame $\Sigma \equiv \mathbf{q}_k$; dr', dt' are similar parameters measured in the moving frame $\Sigma' \equiv \mathbf{q}_{k'}$; ($c = 1$). Equation (60a) holds under the transformation (rotational equation)

$$\Sigma' = O\Sigma, \quad O \in \mathrm{SO}(1,2) \tag{60b}$$

obviously keeping invariant the square of Eq. (60a), the scalar space-time line element $ds^2 = dt^2 - dr^2 = dt'^2 - dr'^2$. Equations (60) are the main equalities of the quaternion relativity theory ("square root" from the special relativity). Since the groups SO(1, 3) and SO(3, ℂ) are 1 : 1 isomorphic, their matrix elements uniquely expressed through each other [28], quaternion relativity must comprise all standard relativistic effects. This is indeed the case.

Boost. Let Σ′ be a result of the simple hyperbolic rotation of Σ

$$\Sigma' = O_3^{i\eta}\Sigma, \quad O_3^{i\eta} = \begin{pmatrix} \cosh\eta & i\sinh\eta & 0 \\ -i\sinh\eta & \cosh\eta & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

then Eq. (60a) yields the standard boost transformations $dr' = dt\sinh\eta + dr\cosh\eta$, $dt' = dt\cosh\eta + dr\sinh\eta$. If Σ′ is observed, then $dr' = 0$, relative velocity is $u = dr/dt = \tanh\eta$.

Addition of velocities. Let frames Σ′ and Σ″ move relatively to Σ so that the angle between their velocities $u_1 = \tanh\eta_1$, $u_2 = \tanh\eta_2$, is β in the plane formed by vectors ($\mathbf{q}_2$, $\mathbf{q}_3$) of Σ, while $\mathbf{u}_1 \uparrow\uparrow \mathbf{q}_2$. Then the rotational equations are $\Sigma' = O_3^{i\eta_1}\Sigma$, $\Sigma'' = O_3^{i\eta_2}O_1^{\beta}\Sigma$; from this system one expresses Σ″ as function of Σ′, $\Sigma'' = O_3^{i\eta_2}O_1^{\beta}O_3^{-i\eta_1}\Sigma'$. Comparison of the first row of this matrix equation with Eq. (60a) written in the form $i\mathbf{q}_{1''} = \cosh\eta(i\mathbf{q}_{1'} + u_y\mathbf{q}_{2'} + u_z\mathbf{q}_{3'})$ (since $dt'/dt'' = \cosh\eta$) yields the standard components and the norm of Σ″ − Σ′ relative velocity

$$u_y = \frac{u_1 - u_2\cos\beta}{1 - \mathbf{u}_1\cdot\mathbf{u}_2}, \quad u_z = -\frac{u_2\sin\beta\sqrt{1-u_1^2}}{1 - \mathbf{u}_1\cdot\mathbf{u}_2}, \quad u^2 = \frac{(u_1-u_2)^2 - (\mathbf{u}_1\times\mathbf{u}_2)^2}{(1-\mathbf{u}_1\cdot\mathbf{u}_2)^2}.$$

If the frames Σ″ and Σ′ move oppositely ($\beta = \pi$) then these equations are reduced to those of parallel motion $u = (u_1 + u_2)/(1 + u_1u_2)$.

Hyperbolic motion. Parameters of quaternion triads may depend on time, then the frames are non-inertial. Let Σ' move relatively to Σ (along $\mathbf{q}_2$) with acceleration $a' =$ const.; the rotational equation is $\Sigma' = O_3^{i\eta(t')}\Sigma$. For the case "$\Sigma'$ observed from Σ" time derivative of Eq. (60a) gives the acceleration $\mathbf{a}' = d^2\mathbf{s}/dt'^2 = id\mathbf{q}_{1'}/dt' = \dot{\eta}\mathbf{q}_2 = a'\mathbf{q}_2$, hence the parameter function is $\eta(t') = a't'$ (no initial velocity). Then from the ratio $dt/dt' = \cosh\eta$ one finds $\Sigma' - \Sigma$ time dependence and all cinematic characteristics: relative velocity, acceleration, and coordinate, all functions the same as in special relativity [29]. But inverse problem (inertial frame Σ observed from accelerated frame Σ') is not considered in special relativity; though it is easily solved in quaternion theory. The inverted rotational equation $\Sigma = O_3^{-i\eta(t')}\Sigma'$ (with the same parameter $\eta(t') = a't'$) gives the ratio $dt'/dt = \cosh\eta$, and the acceleration $\mathbf{a} = d^2\mathbf{s}/dt^2 = 0$ expectedly vanishing. The cinematic characteristics of Σ observed from Σ' are then computed as

$$t(t') = \frac{1}{a'}\arcsin[\tanh(a't')], \quad u(t') = \tanh(a't'),$$

$$r'(t') = \frac{1}{a'}\ln[\cosh(a't')], \quad a(t') = \frac{a'}{\cosh^2(a't')}.$$

For small times the solutions become those of non-relativistic uniformly accelerated rectilinear motion; but for Σ'-observer the Σ clock tends to stop at the time limit $t_{t'\to\infty} \to \pi c/(2a')$.

Thomas precession. Let a constantly oriented frame Σ be at the center of a circular orbit (of radius R) of a frame Σ' oriented identically but orbiting with constant velocity ωR; $\mathbf{q}_1$ is normal to the orbit's plane. For Σ-observer $\mathbf{q}_{2'}$ apparently rotates (Thomas precession [30]). In special relativity the effect is prolongly analyzed [31, 32]. In quaternion theory one just finds projection of $\mathbf{q}_{2'}$ onto $\mathbf{q}_3$ (initially $\mathbf{q}_{2'}\perp\mathbf{q}_3$) from rotational equation $\Sigma' = O_1^{-\gamma(t')}O_2^{i\eta}O_1^{\gamma(t)}\Sigma$ where $\gamma(t) = \omega t$, so that O_1^{γ} makes vector No. 2 of new frame chase Σ'; $\tanh\eta = \omega R =$ const. so that $O_2^{i\eta}$ makes next new frame relativistic; $-\gamma(t') = -\omega' t'$ so that $O_1^{-\gamma(t')}$ makes Σ' constantly oriented. The time-ratio $t' = t/\cosh\psi$ leads to $\Sigma' - \Sigma$ angular velocity relation $\omega' = \omega\cosh\eta$ and to the sought-for projection (the last term in the second row of the rotational equation) $\langle\mathbf{q}_{2'}\rangle_3 \cong \sin(\omega_T t)$ where $\omega_T \equiv \omega - \omega' = \omega(1 - \cosh\eta) = -\frac{\omega}{2}(\frac{\omega R}{c})^2$ is the Thomas precession frequency. Note that quaternion relativity allows computation of this effect on orbits of any shape from any frames (details in [23]).

3.2.2. *New relativistic effects*

Simplicity of the quaternion relativity algorithms prompts to search for new effects and explanations of puzzling observations; few illustrative examples are below.

Relativistic oscillator. Let Σ' move along $\mathbf{q}_{2'}$ under action of a harmonic force, the proper acceleration (force per unit mass) being the function of Σ' time

$\mathbf{a}' = (d\eta/dt')\mathbf{q}_{2'} = \Omega'\beta\cos\Omega' t'\mathbf{q}_{2'}$, Ω' is a proper frequency, $\beta < 1$ is a constant, so the parameter is $\eta(t') = \beta\sin\Omega' t'$. If Σ' is observed from Σ the rotational equation $\Sigma' = O_3^{i\eta(t')}\Sigma$ leads to $\Sigma - \Sigma'$ time interdependence $t = \int\cosh(\beta\sin\Omega' t')dt'$; since $\beta < 1$ the integral is calculated exactly as a series [14]

$$t = t' + \sum_{n=1}^{\infty}\frac{\beta^{2n}}{(2n)!}\left[\frac{1}{2^{2n}}\binom{2n}{n}t' + \frac{(-1)^n}{2^{2n-1}}\sum_{k=0}^{n-1}(-1)^k\binom{2n}{k}\frac{\sin(2n-2k)\Omega' t'}{(2n-2k)\Omega'}\right].$$

On the cycle $\Omega' T' = 2\pi$ one finds $\Sigma - \Sigma'$ period and frequency ratios

$$T = T'\left(1 + \sum_{n=1}^{\infty}\frac{\beta^{2n}}{(2n)!}\frac{1}{2^{2n}}\binom{2n}{n}\right), \quad \Omega = \Omega'\left(1 + \sum_{n=1}^{\infty}\frac{\beta^{2n}}{(2n)!}\frac{1}{2^{2n}}\binom{2n}{n}\right)^{-1},$$

meaning that the oscillation seen from Σ is slower than in reality. Cinematic functions of Σ', velocity, acceleration and coordinate, are computed in approximation $\beta \ll 1$ (while those of the inverse problem are exact functions of Σ'-time). This solution provides good conditions for analysis of the famous "twin paradox" if the cosmic trip is regarded as one oscillation of Σ'. Indeed, not only spatial positions of the twins coincide at the initial and final points, but there the twins are mutually immobile, while the close-to-light speed is achieved gradually (details in [23]). These conditions are not available in special relativity.

Relativistic shift of planets' satellites and explanation of Phobos motion peculiarities. Let a Solar system planet Σ' have a satellite (on orbit of radius R) observed from the Earth Σ; the rotational equation is $\Sigma' = O_3^{i(\mathrm{arctanh}\frac{V}{c})}\Sigma$, the relative $\Sigma' - \Sigma$ velocity $V = \sqrt{V_E^2 + V_P^2 - 2V_E V_P\cos\Omega t}$ expressed through respective (constant) velocities V_P, V_E and the difference $\Omega \sim \mathrm{const.}$ between Σ and Σ' orbital angular velocities, $c \neq 1$. Let $dt' \to T'$ be a real period of the satellite's revolution (small compared to time of observation), $dt \to T$ being the similar "period" observed from Σ. Then the standard time ratio is $T' = T/\cosh\eta = T\sqrt{1 - V^2(t)/c^2}$, i.e. $T > T'$, and the difference between the satellite's real orbital velocity V'_S and observed velocity $V_S(t)$ is $V'_S - V_S(t) = 2\pi R(1/T' - 1/T) = V'_S(A - B\cos\Omega t)$, $A \equiv (V_P^2 + V_E^2)/2c^2 \ll 1$, $B \equiv V_P V_E/c^2 \ll 1$. Time derivative gives the satellite's apparent acceleration $a = \dot{V}_S(t) = -V'_S\Omega B\sin\Omega t$, integration gives its apparent position shift seen from the Earth $\Delta l \equiv \int(V'_S - V_S)dt = V'_S(At - B\sin\Omega t/\Omega)$. Compute these values for Martian satellite Phobos [33] using the data $V_E = 2.978\times10^6\,\mathrm{cm\,s^{-1}}$, mean orbital velocity of Mars is $V_P = 2.413\times10^6\,\mathrm{cm\,s^{-1}}$, Earth–Mars angular velocity difference is $\Omega = 0.932\times10^{-7}\,\mathrm{s^{-1}}$, orbital velocity of Phobos is $V'_S = 2.14\times10^5\,\mathrm{cm\,s^{-1}}$. Then the maximal seen accelerations of Phobos are $a_{\max} = \pm1.59\times10^{-10}\,\mathrm{cm\,s^{-2}} = \pm4.84\times10^{-3}\,\mathrm{deg\,yr^{-2}}$, the experimental values [34–36] well inside the limits. Moreover, after $\sim 9\,\mathrm{yr}$ of the last Earth's observation of the satellite on orbit (in a perihelion opposition) a spacecraft will find Phobos $\sim 5\,\mathrm{km}$ ahead of its expected position, as actually happened [37, 38]. Similar effects are expected to be discovered in fine observations of the Jovian satellites Metis and Adrastea, their visible acceleration estimated as $a_A \leq 5.5 \times 10^{-3}\,\mathrm{deg\,yr}^{-2}$, one degree of an orbit being $\sim$4,000 km.

Relativistic explanation of the Pioneer anomaly. The space probe Pioneer 10 (launched by the USA in 1972 and aimed to leave the Solar system) for more than 20 years was sending signals to the Earth's observers who comparing the data with the Doppler effect expectations discovered the probe's nearly constant deceleration $a_P = -(8 \pm 3) \cdot 10^{-8}$ sm $\cdot$ s^{-2} [39] (similar effect was noticed for Pioneer 11). Many attempts were made to explain the effect by influence of new physical forces, but a simple study [40] demonstrates probable relativistic nature of the anomaly. Let the probe Σ' (mass m) move rectilinearly with velocity $u(t)$ aligned with vector $\mathbf{q}_2$ of frame Σ of the Sun (mass M). The rotational equation $\Sigma' = O_3^{-i\eta}\Sigma$ yields the time ratio $dt = dt' \cosh\eta$, hence frequency $f(t)$ of the signal received in Σ as function of genuine frequency $f' =$ const. of the probe $f(t) = f'\sqrt{1-(u/c)^2} \cong f'(1-(u/c)^2/2)$. So due to the relativistic time contraction effect a relative difference between sent and received signal frequencies emerges

$$\Delta f/f' = \frac{f'-f}{f'} = \frac{u^2}{2c^2} = \kappa/c, \tag{61}$$

here κ is the probe's kinetic energy per mass. But experimentally received value $\Delta f/f'$ may be mistaken for presence of some additional speed u_A responsible for an extra Doppler effect

$$f = \frac{f'}{\sqrt{1-(u_A/c)^2}}\left(1 - \frac{u_A}{c}\cos\gamma\right), \tag{62}$$

here γ is an angle between the probe's velocity and the signal's wave vector (the angle faintly affects the result, so will be ignored, $\gamma = 0$). Then Eq. (62) yields $u_A \cong c\Delta f/f$; comparing this with Eq. (61) one finds the apparent "additional velocity"

$$u_A \cong \kappa/c. \tag{63}$$

In Newtonian mechanics $\kappa = E_0/m + GM/r$, (E_0 is the probe's mechanical energy, G is the gravitational constant,r is $\Sigma - \Sigma'$ distance), so the probe's apparent deceleration is

$$a_A \equiv \dot{u}_A = -uGM/(cr^2).$$

In general relativity (Schwarzschild gravity) the deceleration is shown doubled [41]

$$a_A = -2u\,GM/(cr^2). \tag{64}$$

Insert in Eq. (64) the values $c = 2.99 \times 10^{10}\,\mathrm{cm \cdot s^{-1}}$, $G = 6.67 \times 10^{-8}\,\mathrm{cm^3 \cdot g \cdot s^{-2}}$, $M = 1.99 \times 10^{33}\,\mathrm{g}$, and the NASA data for Pioneer 10 (observed in 1983–1990) $u = (1,52 - 1,41) \times 10^6\,\mathrm{cm \cdot s^{-1}}$, $r = (4.3 - 7.2) \times 10^{14}\,\mathrm{cm}$ [42] to find the deceleration diapason $a_A = -(7.27 - 2.61) \times 10^{-8}\,\mathrm{cm \cdot s^{-2}}$ quite close to the experimental values. But Eq. (64) states that the anomalous acceleration should not be constant, and recent analysis [43] of the observational data confirmed that the Pioneer 10 acceleration indeed decreases as $\Delta a_P = -(0.25 - 0.17) \times 10^{-8}\,\text{cm}\cdot\text{s}^{-2}\cdot\text{yr}^{-1}$. Find respective theoretical value. At the middle stage of observations the $\Sigma - \Sigma'$ distances were [42] $r_{1990} = 7.2 \times 10^{14}\,\text{cm}$, $r_{1991} = 7.58 \times 10^{14}\,\text{cm}$, and Eq. (64) gives a year drop of the acceleration $\Delta a_A = \frac{2uGM}{A}(\frac{1}{r_{1991}^2} - \frac{1}{r_{1990}^2}) = -0.236 \cdot 10^{-8}\,\text{cm}\cdot\text{s}^{-2}\cdot\text{yr}^{-1}$ in full agreement with experiment.

3.2.3. *Dynamics in quaternion relativity*

Thus quaternion relativity appears to be a good tool to deal with cinematic effects; not worse it describes relativistic dynamics. But before turning to the dynamics, picture of the quaternion universe model (not a 4D space-time) is to be commented. Locally the quaternion universe consists of two 3D spaces (separated by the light barrier), one is (the observer's) real 3D space, the other is a 3D space imaginary (for the observer) where a time direction is singled out [14, 23]; the time value is found as ratio of the imaginary space length and the speed of light. This model geometrically reflects the group's SL(3,C) structure and helps to adopt imaginary components arising in the (3+3)D dynamics.

The basic object of the dynamics is the momentum vector. Let two interacting particles (rest masses M_0 and m_0) be attached to frames Σ, Σ'. If Σ' is observed from Σ then ($c \neq 1$)

$$d\mathbf{s} = icdt'\mathbf{q}_{1'} = icdt\mathbf{q}_1 + dr\mathbf{q}_2, \tag{65}$$

$\mathbf{q}_2$ always parallel to relative velocity. Time derivatives of Eq. (65) (with factor m_0), the ratio $dt = dt'\cosh\eta$ taken into account, give the remarkable equality

$$\mathbf{P}' = m_0\frac{d\mathbf{s}}{dt}\frac{dt}{dt'} = m_0\cosh\eta\frac{d\mathbf{s}}{dt} = m\frac{d\mathbf{s}}{dt} \equiv \mathbf{P} \tag{66}$$

($m = m_0\cosh\eta$ as in special relativity); i.e. the Σ' momentum vector (66) (as the interval (65)) is form-invariant under frames $SO(1,2)$ rotations; its components found from Eqs. (65), (66) are

$$\mathbf{P}' = im_0c\mathbf{q}_{1'}, \tag{67a}$$

$$\mathbf{P} = m_0\cosh\eta(ic\mathbf{q}_1 + u\mathbf{q}_2). \tag{67b}$$

Equations (67) allow formulation of the dynamic equation in Newtonian format in Σ and Σ' [44].

Σ' is the base. Then $\partial_{t'}\mathbf{P}' = m_0ic(\Omega_{1'2'}\mathbf{q}_{2'} + \Omega_{1'3'}\mathbf{q}_{3'})$ (Eq. (29) in the form $\partial_{t'}\mathbf{q}_{k'} = \Omega_{k'n'}\mathbf{q}_{n'}$ is used) the connection components being proper tangent $\Omega_{1'2'} = -ia_{2'}/c$ and normal $\Omega_{1'3'} = -ia_{3'}/c$ accelerations of Σ'. Dynamic equation in this case is just definition of the force $F' = m_0a_{k'}\mathbf{q}_{k'}$ or

$$\partial_{t'}\mathbf{P}' = \mathbf{F}'. \tag{68}$$

Equation (68) helps to solve inverse mechanical problems of finding the force if the motion law is known.

Σ is the base. Then

$$\partial_t \mathbf{P} = m_0[\dot{\eta}\sinh\eta(ic\mathbf{q}_1 + u\mathbf{q}_2) \\ + \cosh\eta(ic\Omega_{12}\mathbf{q}_2 + ic\Omega_{13}\mathbf{q}_3 + \dot{u}\mathbf{q}_2 + u\Omega_{21}\mathbf{q}_1 + u\Omega_{23}\mathbf{q}_3)],$$

where $\Omega_{12} = -ia_2/c$, $\Omega_{13} = -ia_3/c$ are proper tangent and normal accelerations of Σ, $\Omega_{23} \equiv \Omega$ is angular velocity of its rotation. The dynamic equation $\partial_t P = \mathbf{F}$ has the components

$$im\frac{u}{c}(c\dot{\eta} + a_2) = iF_1, \tag{69a}$$

$$m(c\dot{\eta} + a_2) = F_2, \tag{69b}$$

$$m(u\Omega + a_3) = F_3. \tag{69c}$$

The system (69) has three specific features.

(1) The first component is in the imaginary 3D space. One may speculate that as the tangent force F_2 pushing the body changes the space scale, the force F_1 similarly changes the time scale; anyway if $F_1 = (u/c)F_2$ then Eq. (69a) exactly repeats Eq. (69b).
(2) Equations (69) are obviously relativistic comprising time derivative of the hyperbolic parameter.
(3) The dynamics automatically takes into account proper accelerations of Σ, the privilege of use of quaternion frames. For $u/c \ll 1$ Eqs. (69) become those of classical mechanics.

3.2.4. *Accelerations and geometry*

The rotational equation $\Sigma' = O_3^{\eta}\Sigma$ establishes the links $\partial_{t'} = \cosh\eta\partial_t$, $\mathbf{q}_{1'} = \cosh\eta\mathbf{q}_1 - i\sinh\eta\mathbf{q}_2$, $\mathbf{q}_{2'} = i\sinh\eta\mathbf{q}_1 + \cosh\eta\mathbf{q}_2$, $\mathbf{q}_{3'} = \mathbf{q}_3$; inserted in Eq. (68) they yield expressions of tangent and normal accelerations of Σ' observed from Σ through the frames proper accelerations

$$c\dot{\eta} = a_{2'} - a_2, \quad V\Omega = \frac{a_{3'}}{\cosh\eta} - a_3. \tag{70}$$

Equations (70) have simple geometrical sense, they are a "physical equivalent" of Eq. (30) stating that the quaternion connection is not a tensor with respect to the frame's transformations. For connection Ω_{kn} this formula is rewritten as $\Omega_{k'n'} = O_{k'j}O_{n'm}\Omega_{jm} + \partial_t O_{k'j}O_{n'j}$, tensor properties spoiled by the last term. In this case of a simple rotation only two components differ from zero, $\Omega_{1'2'} = \Omega_{12} - i\dot{\psi}$, $\Omega_{1'3'} = \cosh\psi\Omega_{13} - i\sinh\psi\Omega_{23}$, insertion of them into the last equalities gives precisely Eqs. (70). Thus Eqs. (68), (69) of relativistic dynamics reflect of geometrical properties of the quaternion space.

3.2.5. *Two-body problem*

Express the acceleration components from Eq. (70) ("seen" from Σ) through a force $a_{2'} = F'_2/m$, $a_2 = F_2/M_0$, $a_{3'} = F'_3/m_0$, $a_3 = F_3/M_0$, where F'_k, F_k are components of the same force acting onto Σ' and Σ, tangent Σ'-acceleration depending on the relativistic mass $m = m_0 \cosh\eta$ of the observed frame Σ', normal acceleration depending on the rest mass; then Eqs. (70) become

$$c\partial_t \eta = F'_2/m - F_2/M_0, \quad V\Omega = F'_3/m - F_3/M_0. \tag{71a}$$

The mechanical system is symmetric so similar equations can be written for Σ observed from Σ'

$$c\partial_{t'} \eta = F_{2'}/M - F'_{2'}/m_0, \quad V\Omega' = F_{3'}/M - F'_{3'}/m_0. \tag{71b}$$

Equations (71) describe relativistic two-body dynamics; they can be used (in similar methodics as equations of Newton dynamics) to search for approximate solutions of problems including effects of interaction and signal retardations or of motions when retardation parameters can be ignored (static or small gradients forces, small $\Sigma' - \Sigma$ distances). The problem of a particle's motion in field of a central force produced by a great mass is an illustrative example [45]. In particular, the solution states that a body rectilinearly moving off the Sun has a deceleration caused by the Galaxy background gravity. If the Galaxy part (comprising the halo) is limited by sphere of the Solar orbit radius $R \sim 2 \times 10^{22}$ cm, then its roughly assessed mass is $M_S \sim 10^{45}$ g [46], and the "Galactic caused deceleration" value of the space probe Pioneer 10 (in 1983) is $a_G \cong -0.44 \times 10^{-8}\,\mathrm{cm} \cdot \mathrm{s}^{-2}$, thus noticeably incorporating to the observed Pioneer anomaly.

Finishing the chapter one has to add that the quaternion relativity basically exploiting 3D vector triads surely deserves reformulation in terms of 2D-cell spinor vectors of the fundamental surface underlying the 3D physical world.

3.3. *Gauge fields in quaternion math*

It is shown below how electromagnetic field equations and hypothetical Yang–Mills field are recognized among quaternion math equalities.

3.3.1. *Electromagnetic field as Cauchy–Riemann type equations*

Many ways to expose the theory of electromagnetic field are known. Maxwell, the theory author, formulated his equations using freshly discovered quaternions. Later vector version of the equations became conventional as well as 4D tensor variant admitting also a formulation by differential forms. Moreover, the electromagnetic field arises as a gauge field in other fields' variational procedures or otherwise Maxwell equations appear in 5D curved space of Kaluza theory. However, these technologies do not deepen insight into the math sense of physical laws.

Following Maxwell turn back to quaternions to find that the theory of functions of quaternion variable naturally comprises an exact analog of the electrodynamic equations [47, 48]. Let $G = G_0(y) + G_n(y)\mathbf{q}_n$ be a continuous and smooth function of the argument $y \equiv y_0 - y_k\mathbf{q}_k$ (all units constant), so that its derivative (e.g. left) can be defined

$$\vec{d}_y G \equiv \frac{\partial G}{\partial y_0} + \mathbf{q}_n \frac{\partial G}{\partial y_n}. \tag{72}$$

To eliminate the ambiguity of the derivative (similar to that of a function of complex variable) a Cauchy–Riemann type condition is suggested $\vec{d}_{\bar{y}} G = 0$ the derivative taken with respect to the conjugate quaternion variable $\bar{y} \equiv y_0 + y_k\mathbf{q}_k$, or explicitly

$$\vec{d}_{\bar{y}} G = \frac{\partial G}{\partial y_0} - \mathbf{q}_n \frac{\partial G}{\partial y_n} = 0. \tag{73}$$

In the "physical case" $G(y) \to A(u) \equiv i\varphi + A_k\mathbf{q}_k$, $y \to u \equiv -ict - x_k\mathbf{q}_k$, with φ, A_k being potentials, x_k, t being space-time coordinates, the differential operator and its conjugate are

$$d_u \equiv \frac{i}{c}\frac{\partial}{\partial t} + \mathbf{q}_n \frac{\partial}{\partial x_n}, \quad d_{\bar{u}} \equiv \frac{i}{c}\frac{\partial}{\partial t} - \mathbf{q}_n \frac{\partial}{\partial x_n}$$

and the derivative of $A(u)$ is

$$\vec{d}_u A \equiv F(u) = -\partial_t\varphi/c - \partial_n A_n + \mathbf{q}_n(i\partial_t A_n/c + i\partial_n\varphi + \varepsilon_{jmn}\partial_j A_m). \tag{74}$$

With the Lorentz gauge $\partial_t\varphi/c + \partial_n A_n = 0$ and the notations $E_n \equiv -\partial_t A_n/c - \partial_n\varphi$, $H_n \equiv \varepsilon_{jmn}\partial_j A_m$ Eq. (74) defines the field intensity vector quaternion $F(u) = (H_n - iE_n)\mathbf{q}_n$. It is a differentiable function if Eq. (73) holds

$$\vec{d}_{\bar{u}} F = i\partial_n E_n - \partial_n H_n - \mathbf{q}_k \left[i\left(\frac{1}{c}\dot{H}_k + \varepsilon_{mnk}\partial_m E_n \right) + \frac{1}{c}\dot{E}_k - \varepsilon_{mnk}\partial_m H_n \right] = 0, \tag{75a}$$

separating in Eq. (75a) scalar, vector, real, and imaginary parts

$$\partial_n E_n = 0, \quad \partial_t E_n/c - \varepsilon_{jmn}\partial_j H_m = 0, \quad \partial_n H_n = 0, \quad \partial_t \dot{H}_n/c + \varepsilon_{jmn}\partial_j E_m = 0, \tag{75b}$$

one gets precisely Maxwell vacuum equations, thus a pure mathematical issue. In Eqs. (75) time is linked with a scalar unit, so Eqs. (75b) are SO(1, 3) covariant, while Eqs. (75a) are not SO(3, C) form-invariant under transformations of quaternion units, so they are not coherent with the quaternion relativity. This fact nonetheless just underlines the difference between approaches to measure time by a cycle-process clock or by a geometric clock (distances per speed of light).

3.3.2. *Yang–Mills field as curvature of a quaternion space*

The Yang–Mills field arises in the localization procedure of spinor field transformations [49, 50] $\psi \to U(y^\beta)\psi$, y^β being 4D space-time coordinates. If a partial derivative of ψ in respective Lagrangian is substituted by the covariant one $\partial_\beta \to D_\beta \equiv \partial_\beta - A_\beta$ {where $A_\beta \equiv iA_{C\beta}\mathbf{T}_C$, $\mathbf{T}_C$ are traceless matrices (a Lie group generators) commuting as $[\mathbf{T}_B, \mathbf{T}_C] = if_{BCD}\mathbf{T}_D$ with structure constants $f_{BCD}\}$, then the matrix of the transformation is constant

$$D_\beta U \equiv (\partial_\beta - A_\beta)U = 0 \tag{76}$$

and the Lagrangian including the term $L_{\rm YM} \sim F^{\alpha\beta}F_{\alpha\beta}$ is invariant, $F_{\alpha\beta} \equiv F_{C\alpha\beta}\mathbf{T}_C$. The gauge field with intensity $F_B^{\mu\nu}$ is expressed through the potentials $A_{B\mu}$ and the structure constants $F_{C\alpha\beta} = \partial_\alpha A_{C\beta} - \partial_\beta A_{C\alpha} + f_{CDE}A_{D\alpha}A_{E\beta}$. Variation of action with the Lagrangian $L_{\rm YM}$ yields the vacuum Yang–Mills equations

$$\partial_\alpha F^{\alpha\beta} + [A_\alpha, F^{\alpha\beta}] = 0. \tag{77}$$

In particular, the group generators can be represented as $i\mathbf{T}_B \to \mathbf{q}_n = -i\boldsymbol{\sigma}_n$, $\boldsymbol{\sigma}_n$ being Pauli matrices, then $f_{BCD} \to \varepsilon_{knm}$, and the potential and intensity are

$$A_\beta = \frac{1}{2}A_{n\beta}\mathbf{q}_n, \tag{78}$$

$$F_{k\alpha\beta} = \partial_\alpha A_{k\beta} - \partial_\beta A_{k\alpha} + \varepsilon_{kmn}A_{m\alpha}A_{n\beta}. \tag{79}$$

But the quaternion format (78) of the Yang–Mills field hardly helps to judge of its alliance with geometry of the hypercomplex math; one can only assume that quaternion spaces contain the field's geometric analog.

Test the assumption considering a 4D space-time with 3D quaternion space section represented by a triad $\mathbf{q}_k$ with the proper connection $\omega_{\alpha mk} \neq 0$, so that the covariant derivative vanishes

$$D_\alpha \mathbf{q}_k \equiv (\delta_{mk}\partial_\alpha + \omega_{\alpha mk})\mathbf{q}_m = 0. \tag{80}$$

Let the triad $\mathbf{q}_k$ be a result of the transformation $\mathbf{q}_k = S(y)\mathbf{q}_{\tilde{k}}S^{-1}(y)$, $\mathbf{q}_{\tilde{k}} = \text{const.}$, $S(y)$ is a variable matrix of the spinor group, then Eq. (80) gives

$$\partial_\alpha S\mathbf{q}_{\tilde{k}}S^{-1} + S\mathbf{q}_{\tilde{k}}\partial_\alpha S^{-1} = \omega_{\alpha kn}S\mathbf{q}_{\tilde{n}}S^{-1},$$

multiplication (from the left) by the combination $\mathbf{q}_{\tilde{k}}S$ leads to the equality

$$\mathbf{q}_{\tilde{k}}S\partial_\alpha S^{-1}\mathbf{q}_{\tilde{k}}S - 3\partial_\alpha S = \omega_{\alpha kn}\mathbf{q}_{\tilde{n}}\mathbf{q}_{\tilde{k}}S. \tag{81}$$

Computation of the first term, product of five non-commutative multipliers, results in remarkably simple expression $\mathbf{q}_{\tilde{k}}S\partial_\alpha S^{-1}\mathbf{q}_{\tilde{k}}S = -\partial_\alpha S$, while the right-hand side is $\omega_{\alpha kn}\mathbf{q}_{\tilde{k}}\mathbf{q}_{\tilde{n}}S = \frac{1}{2}\varepsilon_{knj}\omega_{\alpha kn}\mathbf{q}_j$. With the notation

$$A_\alpha \equiv \frac{1}{2}A_{j\alpha}\mathbf{q}_{\tilde{j}} = \frac{1}{4}\varepsilon_{knj}\omega_{\alpha kn}\mathbf{q}_{\tilde{j}}, \tag{82}$$

Eq. (81) takes the form

$$D_\alpha S \equiv (\partial_\alpha - A_\alpha)S = 0,$$

the same as Eq. (76) but obtained in the way alien to "physical" variational procedures. It is the form-invariance of quaternion multiplication that provides covariant constancy of matrices of the units transformations, and it is the proper connection that is here associated with the Yang–Mills field potential, respective algebraic equality following from Eq. (82)

$$\omega_{\alpha kn} = \varepsilon_{mkn} A_{m\alpha}. \tag{83}$$

The Yang–Mills intensity tensor (79) built out of the "potential" (83) (i.e. intensity contracted with the discriminant tensor, and denoted $R_{mn\alpha\beta}$)

$$R_{mn\alpha\beta} \equiv \varepsilon_{kmn} F_{k\alpha\beta} = \partial_\alpha \omega_{\beta mn} - \partial_\beta \omega_{\alpha mn} + \omega_{\alpha nk}\omega_{\beta km} - \omega_{\beta nk}\omega_{\alpha km}, \tag{84}$$

is straightforwardly identified with the curvature tensor of the quaternion space. But one easily checks up that the curvature (84) is the result of alternation of double partial derivatives of a variable triad's vector, thus being a zero identity. So in a space with proper connection the "Yang–Mills field" has nonzero potential but vanishing intensity.

Consider a quaternion space of general type comprising apart from proper connection $\omega_{\alpha mk}$ also non-metricity $\hat{\sigma}_{\alpha mk}$ which forces a triad rotate disregarding its coordinate dependence

$$\Omega_{\alpha kn}(y) = \omega_{\alpha kn} + \hat{\sigma}_{\alpha kn},$$

in this case, the triad's covariant derivative does not vanish

$$\hat{D}_\alpha \mathbf{q}_k \equiv (\delta_{mk}\partial_\alpha + \Omega_{\alpha mk})\mathbf{q}_m = \hat{\sigma}_{\alpha mk}\mathbf{q}_k.$$

Alternation of double covariant derivatives of a vector determines the curvature tensor

$$\hat{R}_{kn\alpha\beta} = \partial_\alpha \Omega_{\beta kn} - \partial_\beta \Omega_{\alpha kn} + \Omega_{\alpha km}\Omega_{\beta mn} - \Omega_{\alpha nm}\Omega_{\beta mk}. \tag{85}$$

Define the following contractions with the discriminant tensor

$$\hat{F}_{m\alpha\beta} \equiv \frac{1}{2}\varepsilon_{knm}\hat{R}_{kn\alpha\beta}, \quad \hat{A}_\alpha \equiv \frac{1}{2}\hat{A}_{n\alpha}\mathbf{q}_{\tilde{n}} \equiv \frac{1}{4}\varepsilon_{knm}\hat{R}_{kn\alpha\beta}\mathbf{q}_{\tilde{n}}, \tag{86}$$

with these notations Eq. (86) takes the form

$$\hat{F}_{m\alpha\beta} = \partial_\alpha \hat{A}_{m\beta} - \partial_\beta \hat{A}_{m\alpha} + \varepsilon_{knm}\hat{A}_{k\alpha}\hat{A}_{n\beta}, \tag{87}$$

expectedly repeating the definition of the Yang–Mills field intensity (79), in this case not zero. If the non-metricity components are just Ricci coefficients Φ_{jkn} emerging e.g. in Riemannian spaces (see Sec. 2.3.2), then the field (87) has to do with an "additional gravity" since the Lagrangian $L_{\rm YM} \sim \hat{F}_k^{\alpha\beta}\hat{F}_{k\alpha\beta} \sim \hat{R}_{mn}^{\alpha\beta}\hat{R}_{mn\alpha\beta}$ is used in generalizations of the Einstein's theory of gravitation. If the space has only pure quaternion non-metricity σ_{jkn}, analog of Cartan's torsion, then the field (86) is an independent physical entity. Of course, mixture of the two types of non-metricity is feasible. The geometric liaisons (84)–(86) prompt to seek for the Yang–Mills-type equations among equalities inherent in quaternion spaces. In particular, the Bianchi identity

$$\partial_{[\gamma}\hat{R}_{kn\alpha\beta]} + \hat{R}_{mn[\alpha\beta}\Omega_{\gamma]km} + \hat{R}_{km[\alpha\beta}\Omega_{\gamma]nm} = 0, \tag{88}$$

contracted in k, α and n, γ is known to express the Einstein's tensor conservation (only Greek indices are antisymmetrized by square brackets). Instead, perform the first contraction of Eq. (88) with the discriminant tensor replacing curvatures by the intensity from Eq. (86)

$$\frac{1}{2}\varepsilon_{knl}(\partial_{[\gamma}\hat{R}_{kn\alpha\beta]} + \hat{R}_{mn[\alpha\beta}\Omega_{\gamma]km} + \hat{R}_{km[\alpha\beta}\Omega_{\gamma]nm})$$
$$= \partial_{[\gamma}\hat{F}_{l\alpha\beta]} + 2\varepsilon_{knl}\hat{F}_{k[\alpha\beta}\hat{A}_{n\gamma]} = 0$$

and the second contraction in the indices α, γ (also raising free coordinate index)

$$\partial_\alpha \hat{F}_j^{\alpha\beta} + 2\varepsilon_{jkn}\hat{F}_k^{\alpha\beta}\hat{A}_{n\alpha} + \partial_\alpha \hat{F}_j^{\beta\alpha} + 2\varepsilon_{jkn}\hat{F}_k^{\beta\alpha}\hat{A}_{n\alpha} = 0. \tag{89}$$

Of course Eq. (89) remains an identity due to the antisymmetry $\hat{F}_j^{\alpha\beta} = -\hat{F}_j^{\beta\alpha}$. But one notes that the Yang–Mills equation (77) written in the vector quaternion components as

$$\partial_\alpha F_j^{\alpha\beta}\mathbf{q}_j + F_{[k}^{\alpha\beta}A_{n]\alpha}\mathbf{q}_k\mathbf{q}_n = 0 \to \partial_\alpha F_j^{\alpha\beta} + 2\varepsilon_{jkn}F_k^{\alpha\beta}A_{n\alpha} = 0,$$

is precisely equivalent to any "vanishing half" (first two or last two terms) of the identity (89). So the field equation of this gauge field is in a way present in the quaternion geometry. Moreover, the algebra's $SL(2, \mathbb{C})$-form-invariance makes plausible a "Yang–Mills field" with complex-numbered components, the theory deserving a special study.

4. Conclusion

The above review demonstrates that a good number of formulas of famous physical laws can be discovered in the "hypercomplex math-media", among them physical equations originated on the basis of experimental observations (classical mechanics, electrodynamics) or those emerging in a "flash of genius" (theory of relativity, quantum mechanics, Yang–Mills theory). Not all hypercomplex equations precisely coincide with formulas of the conventional laws, e.g. the purely mathematical version of Hamilton–Jacobi equation contains a non-characteristic additional micro-interaction term, while the universe of quaternion relativity predicting same effects as the Einstein's theory turns out to have six dimensions instead of four. But these "corrections" originated in mathematics perhaps do not "spoil the picture", vice versa, may be they make it more accurate improving the points missed in empiric or heuristic versions. Of course only future experiments confirm or reject the assumption. As to the theory of gravitation not included in the above study, there are few doubts that it can be formulated in a curved quaternion space as one of alternative theories of gravity (for recent review, see [51]).

But there is the most essential feature of "reflection of physics" in the hypercomplex numbers math, the opportunity to regard physical entities from the viewpoint of the "pre-geometric" fundamental surface. This approach represents a helpful base for derivation of the "principle" spinor equations of quantum and classical mechanics, and it immanently unites all examples and facts given above. Up to the author's knowledge no systematic analysis based on the "pre-geometric immersion" of physical laws has been ever undertaken, but it seems to be of a certain interest and may turn out fruitful.

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